CYLINDRICALITY AND AUTONOMY OF INTEGRALS AND LAST MULTIPLIERS OF MULTIDIMENSIONAL DIFFERENTIAL SYSTEMS[1]

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Abstract

The conditions of cylindricality and autonomy of first integrals, last multipliers and integral manifolds for linear homogeneous systems of partial differential equations and total differential systems are established.

Key words: total differential system, ordinary differential system, linear homogeneous system of partial differential equations, first integral, last multiplier, partial integral, Painlevé equations.

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Contents

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Introduction

0.1. Generalities

The subject of investigation is a linear homogeneous system of partial differential equations

$$\mathfrak{L}_j(x)\,y = 0, \quad j = 1,\ldots,m, \tag{∂}$$

where $x \in \mathbb{K}^n$, $m \leqslant n$, linear differential operators of first order

$$\mathfrak{L}_j(x) = \sum_{i=1}^{n} u_{ji}(x)\,\partial_{x_i} \quad \text{for all } x \in G, \quad j = 1,\ldots,m, \tag{0.1}$$

have holomorphic coordinates $u_{ji}\colon G \to \mathbb{K}$, $j = 1,\ldots,m$, $i = 1,\ldots,n$, a domain $G \subset \mathbb{K}^n$, $\mathbb{K}$ is a field of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers, and a total differential system

$$dx = X(t,x)\,dt, \tag{TD}$$

where $t \in \mathbb{K}^m$, $x \in \mathbb{K}^n$, $m \leqslant n$, $dt = \operatorname{colon}(dt_1,\ldots,dt_m)$, $dx = \operatorname{colon}(dx_1,\ldots,dx_n)$, $n \times m$ matrix $X(t,x) = \|X_{ij}(t,x)\|$ has holomorphic elements $X_{ij}\colon \Pi \to \mathbb{K}$, $i = 1,\ldots,n$, $j = 1,\ldots,m$, a domain $\Pi \subset \mathbb{K}^{m+n}$. Under $m = 1$ the system (TD) is an ordinary differential system of n-th order.

With a purpose of unambiguous interpretation of the concepts we use, let's formulate the generalities of integrals theory for systems (∂) and (TD) and define the terminology.

Let's consider the operators (0.1) as being not linearly bound on the domain G [5, pp. 105 – 115]. At that we proceed from definition that linear differential operators of first order

are linearly bound on a domain if they are linearly depended over the field $\mathbb{K}$ in every point of this domain [6, pp. 113 – 114].

A holomorphic scalar function $F\colon G' \to \mathbb{K}$ is called a *first integral* on a domain $G' \subset G$ of system (∂) if [1, pp. 35 – 38; 7, pp. 55 – 94]

$$\mathfrak{L}_j F(x) = 0 \quad \text{for all } \ x \in G', \quad j = 1, \ldots, m. \tag{0.2}$$

A holomorphic scalar function $\mu\colon G' \to \mathbb{K}$ is called a *last multiplier* on a domain $G' \subset G$ of system (∂) if [1, pp. 121 – 124]

$$\mathfrak{L}_j\, \mu(x) = \, -\, \mu(x) \operatorname{div} \mathfrak{L}_j(x) \quad \text{for all } \ x \in G', \quad j = 1, \ldots, m. \tag{0.3}$$

We'll say that a holomorphic scalar function $w\colon G' \to \mathbb{K}$ (a manifold $w(x) = 0$) is a *partial integral*[1] on a domain $G' \subset G$ (an *integral manifold*) of system (∂) if

$$\mathfrak{L}_j\, w(x) = \Phi_j(x) \quad \text{for all } \ x \in G', \quad j = 1, \ldots, m, \tag{0.4}$$

where $\Phi_j\colon G' \to \mathbb{K},\ j = 1, \ldots, m,$ are such functions that

$$\Phi_j(x)\big|_{w(x)=0} = 0 \quad \text{for all } \ x \in G', \quad j = 1, \ldots, m. \tag{0.5}$$

The system (∂) is called a *complete* system if the Poisson bracket of any two its operators (0.1) can be represented as a linear combination of this operators [9, p. 117]

$$\big[\mathfrak{L}_j(x), \mathfrak{L}_\zeta(x)\big] = \sum_{l=1}^{m} A_{j\zeta l}(x)\mathfrak{L}_l(x) \quad \text{for all } \ x \in G, \quad j = 1, \ldots, m, \ \zeta = 1, \ldots, m, \tag{0.6}$$

with holomorphic coefficients $A_{j\zeta l}\colon G \to \mathbb{K},\ j = 1, \ldots, m,\ \zeta = 1, \ldots, m,\ l = 1, \ldots, m.$

If the Poisson brackets of operators (0.1) are symmetric, that is,

$$\big[\mathfrak{L}_j(x), \mathfrak{L}_\zeta(x)\big] = \big[\mathfrak{L}_\zeta(x), \mathfrak{L}_j(x)\big] \quad \text{for all } \ x \in G, \quad j = 1, \ldots, m, \ \zeta = 1, \ldots, m, \tag{0.7}$$

then the system (∂) is called a *jacobian* system [7, p. 62].

The symmetry (0.7) of the Poisson brackets of operators (0.1) is equivalent to

$$\big[\mathfrak{L}_j(x), \mathfrak{L}_\zeta(x)\big] = \mathfrak{O} \quad \text{for all } \ x \in G, \quad j = 1, \ldots, m, \ \zeta = 1, \ldots, m, \tag{0.8}$$

where $\mathfrak{O}$ is the null operator. The identity (0.8) is the representation (0.6) with the coefficients $A_{j\zeta l}(x) = 0$ for all $x \in G,\ j = 1, \ldots, m,\ \zeta = 1, \ldots, m,\ l = 1, \ldots, m.$ Therefore a jacobian system (∂) is complete [7, p. 62].

A differential system

$$\partial_{x_j} y = \mathfrak{M}_j(x)\, y, \quad j = 1, \ldots, m, \tag{N∂}$$

where $x \in \mathbb{K}^n,\ m < n,$ linear differential operators of first order

$$\mathfrak{M}_j(x) = \sum_{s=m+1}^{n} u_{js}(x)\, \partial_{x_s} \quad \text{for all } \ x \in G, \quad j = 1, \ldots, m,$$

have holomorphic coefficients $u_{js}\colon G \to \mathbb{K},\ j = 1, \ldots, m,\ s = m+1, \ldots, n,$ is called a *normal* linear homogeneous system of partial differential equations [7, p. 64].

Let's note that a complete normal system is jacobian [7, p. 65].

The complete system (∂) by means of linear nonsingular on the domain G change of operators (0.1) can be reduced to the complete normal system (at that only the restriction of the domain G may happen) [7, p. 66].

[1] For example, the term *second integral* is used and there is also another terminology (see [8]).

Let the complete system (∂) be such a system that the square matrix $\widehat{u}$ of order m, which is formed by m first columns of $m\times n$ matrix $u(x)=\|u_{ji}(x)\|$ for all $x\in G$, is nonsingular on the domain G. Then, the complete system (∂) can be reduced to the complete normal system in the form of $(\mathrm{N}\partial)$, at that, in the neighbourhood of any point x from the domain G, where $\det\widehat{u}(x)\neq 0$, this systems are integral equivalent [1, pp. 47 – 48].

We'll call a subdomain H of the domain G a *normalization domain* of system (∂) if this system in a neighbourhood of any point of the domain H can be reduced to the integral equivalent complete normal system [1, p. 48].

A set of functionally independent on a domain $G'\subset G$ first integrals $F_l\colon G'\to\mathbb{K}$, $l=1,\dots,k$, of system (∂) is called a *basis of first integrals* (or an *integral basis*) on the domain G' of system (∂) if any first integral $\Psi\colon G'\to\mathbb{K}$ of this system can be represented as $\Psi(x)=\Phi\bigl(F_1(x),\dots,F_k(x)\bigr)$ for all $x\in G'$, where Φ is some holomorphic function on the codomain of vector function $F\colon x\to\bigl(F_1(x),\dots,F_k(x)\bigr)$ for all $x\in G'$. At that, the number k is called the *dimension* of the basis of first integrals on the domain G' of system (∂) [1, p. 38; 7, p. 70; 10, pp. 523 – 525].

A complete linear homogeneous system of partial differential equations (∂) on a neighbourhood of any point from its normalization domain has a basis of first integrals of the dimension $n-m$ [1, p. 51].

Every incomplete system (∂) on a domain G can be reduced to an integral equivalent complete system [7, pp. 243 – 245].

We'll call a number δ a *defect* of an incomplete system (∂) if this system on the domain G can be reduced to an integral equivalent complete system by addition of δ equations as

$$\bigl[\mathfrak{L}_{j_\nu}(x),\mathfrak{L}_{l_\mu}(x)\bigr]\,y=0,\quad \Bigl[\mathfrak{L}_{\alpha_\xi}(x),\bigl[\mathfrak{L}_{j_\nu}(x),\mathfrak{L}_{l_\mu}(x)\bigr]\Bigr]\,y=0,$$

$$\Bigl[\mathfrak{L}_{\beta_\zeta}(x),\Bigl[\mathfrak{L}_{\alpha_\xi}(x),\bigl[\mathfrak{L}_{j_\nu}(x),\mathfrak{L}_{l_\mu}(x)\bigr]\Bigr]\Bigr]\,y=0,\ \dots\,,$$

$\nu=1,\dots,m_1,\ \mu=1,\dots,m_2,\ \xi=1,\dots,m_3,\ \zeta=1,\dots,m_4,\ \dots\,,\ m_s\leqslant m,\ s=1,2,\dots\,,$ $\{1,\dots,m\}\ni j_\nu,l_\mu,\alpha_\xi,\beta_\zeta,\dots$ [1, pp. 42 – 43].

Let's agree on a complete system has the defect $\delta=0$. Then, one can say that every system (∂) has the defect δ and $0\leqslant\delta\leqslant n-m$.

The system (∂) with defect δ on a neighbourhood of any point from its normalization domain has a basis of first integrals of dimension $n-m-\delta$ [1, p. 51].

The system (∂) is complete if and only if on a neighbourhood of every point from its any normalization domain it has a basis of first integrals of dimension $n-m$.

The system (TD) is called *completely solvable* on the domain $\Pi'\subset\Pi$ if in every point $(t_0,x_0)\in\Pi'$ for system (TD) a solution to the Cauchy problem with initial conditions (t_0,x_0) is unique [1, p. 17]. In case, when $\Pi'=\Pi$, we'll say that system (TD) is completely solvable.

The system (TD) is completely solvable if and only if the Frobenius conditions [1, pp. 17 – 25; 11, pp. 290 – 302] are satisfied:

$$\partial_{t_j}X_{i\zeta}(t,x)+\sum_{\xi=1}^{n}X_{\xi j}(t,x)\partial_{x_\xi}X_{i\zeta}(t,x)=\partial_{t_\zeta}X_{ij}(t,x)+\sum_{\xi=1}^{n}X_{\xi\zeta}(t,x)\partial_{x_\xi}X_{ij}(t,x)$$
$$\text{for all }\ (t,x)\in\Pi,\quad i=1,\dots,n,\ j=1,\dots,m,\ \zeta=1,\dots,m. \tag{0.9}$$

The system (TD) induces m linear differential operators of first order

$$\mathfrak{X}_j(t,x)=\partial_{t_j}+\sum_{i=1}^{n}X_{ij}(t,x)\partial_{x_i}\quad\text{for all }\ (t,x)\in\Pi,\quad j=1,\dots,m. \tag{0.10}$$

Then, the Frobenius conditions (0.9) via the Poisson brackets can be written as the system of operator identities

$$\big[\mathfrak{X}_j(t,x), \mathfrak{X}_\zeta(t,x)\big] = \mathfrak{O} \quad \text{for all } (t,x) \in \Pi, \quad j = 1,\dots,m, \ \zeta = 1,\dots,m. \tag{0.11}$$

The system (TD) is the Pfaff system of equations

$$\omega_i(t,x) = 0, \quad i = 1,\dots,n, \tag{0.12}$$

with linear differential forms

$$\omega_i(t,x) = dx_i - \sum_{j=1}^{m} X_{ij}(t,x)\, dt_j \quad \text{for all } (t,x) \in \Pi, \quad i = 1,\dots,n, \tag{0.13}$$

which have holomorphic coefficients $X_{ij}\colon \Pi \to \mathbb{K}$, $i = 1,\dots,n$, $j = 1,\dots,m$.

The Frobenius conditions (0.9) of the complete solvability of system (TD) (the Frobenius conditions of closure of the Pfaff system of equations (0.12)) [11, pp. 299 – 301; 12, pp. 91; 13] can be written via differential 1-forms (0.13) as the system of exterior differential identities

$$d\omega_i(t,x) \wedge \Big(\mathop{\wedge}_{\xi=1}^{n} \omega_\xi(t,x)\Big) = 0 \quad \text{for all } (t,x) \in \Pi, \quad i = 1,\dots,n.$$

A holomorphic scalar function $F\colon \Pi' \to \mathbb{K}$ is called a *first integral* on the domain $\Pi' \subset \Pi$ of system (TD) if the differential of the function F by virtue of system (TD) vanishes identically on the domain Π', that is,

$$dF(t,x)\big|_{\text{(TD)}} = 0 \quad \text{for all } (t,x) \in \Pi'. \tag{0.14}$$

By means of linear differential operators (0.10) the identity (0.14) can be written as the system of identities [1, p. 26]

$$\mathfrak{X}_j F(t,x) = 0 \quad \text{for all } (t,x) \in \Pi', \quad j = 1,\dots,m. \tag{0.15}$$

A holomorphic scalar function $F\colon \Pi' \to \mathbb{K}$ is a first integral on the domain $\Pi' \subset \Pi$ of the completely solvable system (TD) if and only if this function keeps a constant value along any solution $x\colon t \to x(t)$ for all $t \in T \subset \mathbb{K}^m$ to system (TD) such that $(t, x(t)) \in \Pi'$ for all $t \in T$, that is,

$$F(t, x(t)) = C \quad \text{for all } t \in T, \quad C = \text{const}.$$

A holomorphic scalar function $\mu\colon \Pi' \to \mathbb{K}$ is called a *last multiplier* on the domain $\Pi' \subset \Pi$ of system (TD) if [1, pp. 129 – 131; 9, pp. 117 – 130]

$$\mathfrak{X}_j\, \mu(t,x) = -\,\mu(t,x)\, \operatorname{div} \mathfrak{X}_j(t,x) \quad \text{for all } (t,x) \in \Pi', \quad j = 1,\dots,m. \tag{0.16}$$

We'll call a holomorphic scalar function $w\colon \Pi' \to \mathbb{K}$ (a manifold $w(t,x) = 0$) a *partial integral* on the domain $\Pi' \subset \Pi$ (an *integral manifold*) of system (TD) if [1, pp. 161 – 163]

$$\mathfrak{X}_j\, w(t,x) = \Phi_j(t,x) \quad \text{for all } (t,x) \in \Pi', \quad j = 1,\dots,m, \tag{0.17}$$

where $\Phi_j\colon \Pi' \to \mathbb{K}$, $j = 1,\dots,m$, are such the functions that

$$\Phi_j(t,x)\big|_{w(t,x)=0} = 0 \quad \text{for all } (t,x) \in \Pi', \quad j = 1,\dots,m. \tag{0.18}$$

A holomorphic scalar function $w\colon \Pi' \to \mathbb{K}$ (a manifold $w(t,x) = 0$) is a partial integral on the domain $\Pi' \subset \Pi$ (an integral manifold) of the completely solvable system (TD) if and only if the function w vanishes identically along any solution $x\colon t \to x(t)$ for all $t \in T \subset \mathbb{K}^m$ to system (TD) such that $(t, x(t)) \in \Pi'$ for all $t \in T$, that is,

$$w(t, x(t)) = 0 \quad \text{for all } t \in T.$$

If a last multiplier μ of system (∂) (of system (TD)) defines a manifold $\mu=0,$ then it is a partial integral of this system.

Indeed, the system of identities (0.3) is the system of identities (0.4), where $\Phi_j(x) = -\mu(x)\operatorname{div}\mathfrak{L}_j(x)$ for all $x\in G'$, $j=1,\ldots,m,$ and therefore the conditions (0.5) are satisfied. Similarly, the system of identities (0.16) is the system of identities (0.17) with the property (0.18).

If a last multiplier μ of system (∂) (of system (TD)) defines a manifold $1/\mu=0,$ then it is an integral manifold of this system.

A set of functionaly independent on the domain $\Pi'\subset\Pi$ first integrals $F_l\colon\Pi'\to\mathbb{K}$, $l=1,\ldots,k,$ of system (TD) is called a *basis of first integrals* (or an *integral basis*) on the domain Π' of system (TD) if any first integral $\Psi\colon\Pi'\to\mathbb{K}$ of this system can be represented as $\Psi(t,x)=\Phi\bigl(F_1(t,x),\ldots,F_k(t,x)\bigr)$ for all $(t,x)\in\Pi'$, where Φ is some holomorphic function on the codomain of vector function $F\colon(t,x)\to\bigl(F_1(t,x),\ldots,F_k(t,x)\bigr)$ for all $(t,x)\in\Pi'$. At that, the number k is called the *dimension* of the basis of first integrals on the domain Π' of system (TD) [1, p. 29].

A completely solvable system (TD) on a neighbourhood of any point from the domain Π has a basis of first integrals of dimension n [1, p. 34].

To construct a basis of first integrals for system (TD) without taking into consideration a solvability of Cauchy problem we'll be based on integrals theory of linear homogeneous system of partial differential equations.

A normal linear homogeneous system of partial differential equations

$$\mathfrak{X}_j(t,x)\,y=0,\quad j=1,\ldots,m, \tag{0.19}$$

is associated to the total differential system (TD).

Directly from definitions of the first integral both for the total differential system and for the linear homogeneous system of partial differential equations we establish the basic integral connection between the systems (TD) and (0.19) [1, pp. 53 – 56].

A holomorphic function $F\colon(t,x)\to F(t,x)$ for all $(t,x)\in\Pi'$ is a first integral on the domain $\Pi'\subset\Pi$ of system (TD) if and only if it is a first integral on the domain Π' of system (0.19) which is associated to system (TD).

In accordance with this property the systems (TD) and (0.19) have locally common basis of first integrals of quite concrete dimension [14].

A total differential system (TD) in a neighbourhood of every point from normalization domain of the associated to this system normal linear homogeneous system of partial differential equations (0.19) has a basis of first integrals of dimension $n-\delta$, where δ is the defect of system (0.19), $0\leqslant\delta\leqslant n$.

0.2. Problem definition

Ordinary differential system of n-th order

$$\frac{dx}{dt}=f(t,x), \tag{D}$$

where $t\in\mathbb{K}$, $x\in\mathbb{K}^n$, $\dfrac{dx}{dt}=\operatorname{colon}\Bigl(\dfrac{dx_1}{dt},\ldots,\dfrac{dx_n}{dt}\Bigr)$, the vector function $f(t,x)=\operatorname{colon}(f_1(t,x),\ldots,f_n(t,x))$ has holomorphic coordinates $f_i\colon\Pi\to\mathbb{K}$, $i=1,\ldots,n,$ the domain $\Pi\subset\mathbb{K}^{n+1}$, has a basis of first integrals of dimension n [15, pp. 156 – 159; 16, pp. 256 – 263].

Autonomous ordinary differential system of n-th order

$$\frac{dx}{dt}=f(x), \tag{AD}$$

where $t \in \mathbb{K}$, $x \in \mathbb{K}^n$, $\dfrac{dx}{dt} = \text{colon}\Big(\dfrac{dx_1}{dt}, \ldots, \dfrac{dx_n}{dt}\Big)$, the vector function $f(x) = \text{colon}\,(f_1(x), \ldots, f_n(x))$ has holomorphic coordinates $f_i\colon G \to \mathbb{K}$, $i = 1, \ldots, n$, the domain $G \subset \mathbb{K}^n$, in the n-dimensional basis has $n-1$ autonomous first integrals which are functionaly independent on the domain G [17, pp. 161 – 169].

V.I. Mironenko has studied [18] whether nonautonomous system (D) can have autonomous first integrals. The possibility of existence of autonomous partial integrals and autonomous last multipliers for the nonautonomous system (D) was studied in [2]. Moreover, in that article I solved the problem whether both the system (D) and the system (AD) can have the first integrals, the last multipliers and the partial integrals which are the functions depending on several r variables $x_{\xi_\tau}, \xi_\tau \in \{1, \ldots, n\}, \tau = 1, \ldots, r, 1 \leqslant r \leqslant n$, but not necessarily depending on all variables x_i, $i = 1, \ldots, n$.

Let's give specific example in which the existence of the autonomous partial integral depending on $r < n$ variables is useful in stability theory.

The nonautonomous real ordinary differential system of third order

$$\frac{dx}{dt} = -x\big(x^2 + (2 + \sin t)y^2\big) - y(e^t x + t^2 xz + ty^2),$$

$$\frac{dy}{dt} = x(e^t x + t^2 xz + ty^2) - y\big(x^2 + (2 + \sin t)y^2\big),$$

$$\frac{dz}{dt} = z\big(x^2 + (2 + \sin t)y^2 + z^2\big)$$

has the autonomous partial integral $w\colon (x, y, z) \to x^2 + y^2$ for all $(x, y, z) \in \mathbb{R}^3$. This partial integral specifies the isolated point $x = y = 0$ in the subspace $\mathbb{R}^2$ of the phase space $\mathbb{R}^3$ and satisfies the hypotheses of Rumyanzev's theorem [19, p. 29]. Therefore the zero solution $x = y = z = 0$ to this system is asymptotically stable with respect to (x, y). At the same time, this solution is unstable by Chetaev's theorem [19, pp. 19 – 20] (one should take $V(x, y, z) = -x^2 - y^2 + z^2$)

Let's define the problem of existence of first integrals, last multipliers and partial integrals for system (∂) which are the functions depending on several variables x_i, $i = 1, \ldots, n$, and for system (TD) which are the functions depending on several independend variables t_j, $j = 1, \ldots, m$, and several dependend variables x_i, $i = 1, \ldots, n$. At that, we'll use the terms and theoretical foundations from introduction.

1. Cylindricality and autonomy of first integrals

1.1. Cylindricality of first integrals for linear homogeneous system of partial differential equations

Definition 1.1. *We'll say that a first integral F on a domain $G' \subset G$ of system (∂) is* ***$(n-k)$-cylindrical*** *if the function F depends only on k, $0 \leqslant k \leqslant n$, variables $x_1, \ldots, x_n$.*

Let's define the problem of existence for system (∂) a $(n-k)$-cylindrical first integral

$$F\colon x \to F({}^k x) \quad \text{for all } x \in G' \subset G, \quad {}^k x = (x_1, \ldots, x_k). \tag{1.1}$$

1.1.1. Necessary condition of existence of cylindrical first integral. According to the definition of first integral, the function (1.1) will be the first integral on the domain $G' \subset G$ of system (∂) if and only if

$${}^k\mathfrak{L}_j F({}^k x) = 0 \quad \text{for all } x \in G', \quad j = 1, \ldots, m, \tag{1.2}$$

where the linear differential operators of first order

$$ {}^k\mathfrak{L}_j(x) = \sum_{\xi=1}^{k} u_{j\xi}(x)\,\partial_{x_\xi} \quad \text{for all } \ x \in G, \quad j = 1, \dots, m. \tag{1.3}$$

Concerning the sets of functions ${}^kU_j = \{u_{j1}(x), \dots, u_{jk}(x)\}$, $j = 1, \dots, m$, the system of identeties (1.2) means that the functions of each set ${}^kU_j$, $j = 1, \dots, m$, are linearly dependent with respect to variable x_p on the domain G' under any fixed values of variables x_i, $i = 1, \dots, n$, $i \neq p$. It holds true under each fixed index $p = k+1, \dots, n$. Therefore the Wronskians of each set ${}^kU_j$, $j = 1, \dots, m$, with respect to variables x_p, $p = k+1, \dots, n$, vanish identically on the domain G', that is, the system of identities

$$ \mathrm{W}_{x_p}\big({}^ku^j(x)\big) = 0 \quad \text{for all } \ x \in G', \quad j = 1, \dots, m, \ p = k+1, \dots, n, \tag{1.4}$$

is satisfied. Here vector functions ${}^ku^j \colon x \to \big(u_{j1}(x), \dots, u_{jk}(x)\big)$ for all $x \in G$, $j = 1, \dots, m$, and W_{x_p} are the Wronskians with respect to x_p, $p = k+1, \dots, n$.

So the *necessary test of existence of* $(n-k)$*-cylindrical first integral for linear homogeneous system of partial differential equations* is proved.

Theorem 1.1. *The system of identities* (1.4) *is a necessary condition of existence of* $(n-k)$*-cylindrical first integral* (1.1) *for system* (∂).

1.1.2. Criterion of existence of cylindrical first integral. Let $m \times n$ matrix $u(x) = \|u_{ji}(x)\|$ for all $x \in G$ of system (∂) satisfies the conditions (1.4). Let's write the functional system

$$ \begin{gathered} {}^ku^j(x)\,{}^k\varphi = 0, \ j = 1, \dots, m, \quad \partial^{\xi}_{x_p}\ {}^ku^j(x)\,{}^k\varphi = 0, \\ j = 1, \dots, m, \quad p = k+1, \dots, n, \quad \xi = 1, \dots, k-1, \end{gathered} \tag{1.5}$$

where a vector function ${}^k\varphi \colon x \to \big(\varphi_1({}^kx), \dots, \varphi_k({}^kx)\big)$ for all $x \in G'$ is unknown, vector functions ${}^ku^j \colon x \to \big(u_{j1}(x), \dots, u_{jk}(x)\big)$ for all $x \in G$, $j = 1, \dots, m$. Let's introduce a Pfaffian equation

$$ {}^k\varphi({}^kx)\, d\ {}^kx = 0. \tag{1.6}$$

Theorem 1.2 (*criterion of existence of* $(n-k)$*-cylindrical first integral for linear homogeneous system of partial differential equations*). *For system* (∂) *to have* $(n-k)$*-cylindrical first integral* (1.1) *it is necessary and sufficient that there exists a vector function* ${}^k\varphi$, *satisfying functional system* (1.5), *such that the function* (1.1) *is the general integral of the Pfaffian equation* (1.6) *on the domain* $\widetilde{G}^k$ *which is the natural projection of domain* G' *on coordinate subspace* $O\ {}^kx$.

Proof. Necessity. Let system (∂) has $(n-k)$-cylindrical first integral (1.1) on the domain G'. Then the identities (1.2) are satisfied:

$$ \sum_{\xi=1}^{k} u_{j\xi}(x)\,\partial_{x_\xi} F({}^kx) = 0 \quad \text{for all } \ x \in G', \quad j = 1, \dots, m. $$

By differentiating this identities $k-1$ times with respect to x_p, $p = k+1, \dots, n$, we conclude that an extension on the domain G' of the function

$$ {}^k\varphi \colon {}^kx \to \big(\partial_{x_1}F({}^kx), \dots, \partial_{x_k}F({}^kx)\big) \quad \text{for all } \ {}^kx \in \widetilde{G}^k $$

is a solution to the functional system (1.5). From this it also follows that the function (1.1) is a general integral on the domain $\widetilde{G}^k \subset \mathbb{K}^k$ of the Pfaffian equation (1.6).

Sufficiency. Let the vector function ${}^k\varphi \colon x \to {}^k\varphi({}^kx)$ for all $x \in G'$ be the solution to the functional system (1.5) and the Pfaffian equation (1.6) which is constructed on its base has a general integral (1.1) on the domain $\widetilde{G}^k \subset \mathbb{K}^k$. Then, the system of identities

$$\partial_{x_\xi} F({}^k x) - \mu({}^k x)\,\varphi_\xi({}^k x) = 0 \quad \text{for all } \ {}^k x \in \widetilde{G}^k, \ \ \xi = 1, \ldots, k, \tag{1.7}$$

is satisfied, where $\mu\colon {}^k x \to \mu({}^k x)$ for all ${}^k x \in \widetilde{G}^k$ is a holomorphic integrating multiplier of the Pfaffian equation (1.6) which corresponds to its general integral (1.1) on the domain $\widetilde{G}^k$.

Hence, taking into account that the function ${}^k\varphi$ is the solution to the functional system (1.5) we receive the system of identities (1.2). Therefore the function (1.1) is an $(n-k)$-cylindrical first integral on the dimain G' of system (∂). ■

Example 1.1. Let's consider a linear homogeneous system of partial differential equations

$$\mathfrak{L}_1(x)y = 0, \quad \mathfrak{L}_2(x)y = 0, \tag{1.8}$$

which is constructed on the base of being not holomorphically linearly bound on the space $\mathbb{K}^4$ linear differential operators of first order

$$\mathfrak{L}_1(x) = x_1x_2\partial_{x_1} - x_1^2\partial_{x_2} + (x_1 + x_2^2 + x_3^2 - x_4^2)\partial_{x_3} + (x_3^2 + x_4^2)\partial_{x_4} \quad \text{for all } \ x \in \mathbb{K}^4,$$

$$\mathfrak{L}_2(x) = x_2^2\partial_{x_1} - x_1x_2\partial_{x_2} + (x_3^2 + x_4^2)\partial_{x_3} + (x_1 + x_2^2 + x_3^2 + x_4^2)\partial_{x_4} \quad \text{for all } \ x \in \mathbb{K}^4.$$

Let's find for system (1.8) a 2-cylindrical first integral

$$F\colon x \to F(x_1, x_2) \quad \text{for all } \ x \in G' \subset \mathbb{K}^4. \tag{1.9}$$

The Wronskians of the sets of functions ${}^2U_1 = \{x_1x_2, -x_1^2\}$ and ${}^2U_2 = \{x_2^2, -x_1x_2\}$ with respect to x_3 and x_4 vanish identically on the space $\mathbb{K}^4$. Therefore the necessary conditions (Theorem 1.1) of existence of 2-cylindrical first integral (1.9) for system (1.8) are satisfied.

Let's write the functional system (1.5):

$$x_1x_2\,\varphi_1 + (\,-x_1^2)\varphi_2 = 0, \quad x_2^2\,\varphi_1 + (\,-x_1x_2)\varphi_2 = 0,$$

$$\partial_{x_3}(x_1x_2)\,\varphi_1 + \partial_{x_3}(\,-x_1^2)\varphi_2 = 0, \quad \partial_{x_3}x_2^2\,\varphi_1 + \partial_{x_3}(\,-x_1x_2)\varphi_2 = 0,$$

$$\partial_{x_4}(x_1x_2)\,\varphi_1 + \partial_{x_4}(\,-x_1^2)\varphi_2 = 0, \quad \partial_{x_4}x_2^2\,\varphi_1 + \partial_{x_4}(\,-x_1x_2)\varphi_2 = 0.$$

From this we get the system

$$x_1x_2\,\varphi_1 - x_1^2\,\varphi_2 = 0, \quad x_2^2\,\varphi_1 - x_1x_2\,\varphi_2 = 0$$

and receive from it the equation

$$x_2\,\varphi_1 - x_1\,\varphi_2 = 0.$$

The solution on the space $\mathbb{K}^4$ to this equation are, for example, the functions

$$\varphi_1\colon x \to x_1 \quad \text{for all } \ x \in \mathbb{K}^4 \qquad \text{and} \qquad \varphi_2\colon x \to x_2 \quad \text{for all } \ x \in \mathbb{K}^4.$$

Let's write on the base of this solution the Pfaffian equation

$$x_1\,dx_1 + x_2\,dx_2 = 0,$$

which has the general integral $F\colon (x_1, x_2) \to x_1^2 + x_2^2$ for all $(x_1, x_2) \in \mathbb{K}^2$.

Therefore the system (1.8) on the space $\mathbb{K}^4$ has the 2-cylindrical first integral

$$F\colon x \to x_1^2 + x_2^2 \quad \text{for all } \ x \in \mathbb{K}^4. \tag{1.10}$$

Since the Poisson bracket

$$[\mathfrak{L}_1(x), \mathfrak{L}_2(x)] = \,-x_2(x_1^2 + x_2^2)\partial_{x_1} + x_1(x_1^2 + x_2^2)\partial_{x_2} +$$

$$+\,(x_1 + 2x_1x_4 - x_3^2 - x_4^2 + 2x_1x_2^2 + 2x_2^2x_4 + 4x_3^2x_4 - 2x_3x_4^2 + 4x_4^3)\partial_{x_3} +$$

$$+\,(x_1x_2+2x_1x_3-2x_1x_4-2x_3^2-2x_1^2x_2+2x_2^2x_3-2x_2^2x_4+2x_3^3-4x_3x_4^2)\partial_{x_4} \quad \text{for all } \ x\in\mathbb{K}^4$$

is not a linear combination on the space $\mathbb{K}^4$ of operators $\mathfrak{L}_1$ and $\mathfrak{L}_2$ the system (1.8) is not complete. Then, a basis of first integrals of system (1.8) consists of no more than one first integral (accurate within the functional expression).

Thus the 2-cylindrical first integral (1.10) of system (1.8) forms its integral basis on the space $\mathbb{K}^4$.

1.1.3. Functionally independent cylindrical first integrals.

Theorem 1.3. *Let the functional system* (1.5) *has* q *not linearly bound on the domain* $G'\subset G$ *solutions*

$$^k\varphi^\gamma\colon x\to {}^k\varphi^\gamma({}^kx) \quad \text{for all } \ x\in G', \quad \gamma=1,\dots q, \tag{1.11}$$

where the vector $^k\varphi^\gamma=(\varphi_1^\gamma,\dots,\varphi_k^\gamma)$, *and the Pfaffian equations*

$$^k\varphi^\gamma({}^kx)\,d\,{}^kx=0, \quad \gamma=1,\dots,q, \tag{1.12}$$

which are constructed on the base of this solutions have correspondingly general integrals

$$F_\gamma\colon {}^kx\to F_\gamma({}^kx) \quad \text{for all } \ {}^kx\in\widetilde{G}^k\subset\mathbb{K}^k, \quad \gamma=1,\dots,q, \tag{1.13}$$

on the domain $\widetilde{G}^k$ *which is the natural projection of domain* G' *on coordinate subspace* $O\,{}^kx$. *Then, this general integrals are functionally independent on the domain* $\widetilde{G}^k$.

Proof. By virtue of the system of identities (1.7)

$$\partial_{x_\xi}F_\gamma({}^kx)=\mu_\gamma({}^kx)\,\varphi_\xi^\gamma({}^kx) \quad \text{for all } \ {}^kx\in\widetilde{G}^k, \quad \xi=1,\dots,k, \ \gamma=1,\dots,q.$$

Therefore the Jacobi's matrix

$$J\big(F_\gamma({}^kx);\,{}^kx\big)=\big\|\mu_\gamma({}^kx)\,\varphi_\xi^\gamma({}^kx)\big\|_{q\times k} \ \text{for all } \ {}^kx\in\widetilde{G}^k.$$

Since the vector functions (1.11) are not linearly bound on the domain $\widetilde{G}^k$ the rank of Jacobi's matrix $\operatorname{rank} J\big(F_\gamma({}^kx);\,{}^kx\big)=q$ for all ^kx from the domain $\widetilde{G}^k$ perhaps with the exception of point set of k-dimensional zero measure. So the general integrals (1.13) of the Pfaffian equations (1.12) are functionally independent on the domain $\widetilde{G}^k$. ■

The Theorem 1.3 (taking into account the Theorem 1.2) let us to find a quantity of functionally independent $(n-k)$-cylindrical first integrals of system (∂).

1.2. First integrals of s-nonautonomous completely solvable total differential systems

Definition 1.2. *We'll say that system* (TD) *is* ***s-nonautonomous*** *if all functions-elements* $X_{ij}\colon\Pi\to\mathbb{K},\ i=1,\dots,n,\ j=1,\dots,m,$ *of the matrix* X *depend on* x *and only on* $s,\ 0\leqslant s\leqslant m,$ *independent variables* $t_1,\dots,t_m$.

Without loss of generality we'll consider that the s-nonautonomous system (TD) has such the functions-elements $X_{ij}\colon\Pi\to\mathbb{K},\ i=1,\dots,n,\ j=1,\dots,m,$ of matrix X that depend only on x and on first s independent variables $t_1,\dots,t_s$, that is,

$$dx=X({}^st,x)\,dt, \tag{TDs}$$

where $^st=(t_1,\dots,t_s)$, $dt=\operatorname{colon}(dt_1,\dots,dt_m)$, $dx=\operatorname{colon}(dx_1,\dots,dx_n)$, $n\times m$ matrix $X({}^st,x)=\|X_{ij}({}^st,x)\|$ has holomorphic elements $X_{ij}\colon({}^st,x)\to X_{ij}({}^st,x)\ \ \forall({}^st,x)\in\Pi^{s+n}$, $i=1,\dots,n,\ j=1,\dots,m,$ the domain $\Pi^{s+n}\subset\mathbb{K}^{s+n},\ 0\leqslant s\leqslant m\leqslant n$.

Under $s=0$ the system (TDs) is the autonomous system (ATD).

Definition 1.3. *We'll say that a first integral F on a domain $\Pi' \subset \Pi$ of system* (TD) *is **s-nonautonomous** if the function F depends on x and only on s, $0 \leqslant s \leqslant m$, independent variables $t_1, \ldots, t_m$.*

If $s = 0$, then an s-nonautonomous first integral of system (TD) is an autonomous first integral of system (TD).

Let ${}^sX$ be the matrix which is formed from the $n \times m$ matrix $X({}^st, x)$ by deletion of first s columns.

Theorem 1.4. *If the rank of matrix ${}^sX({}^st, x)$ of the completely solvable system* (TDs) *is equal to r on the domain Π^{s+n}, then it has on this domain exactly $n - r$ functionally independent s-nonautonomous first integrals $F_\gamma \colon \Pi^{s+n} \to \mathbb{K}$, $\gamma = 1, \ldots, n - r$.*

Proof. Let $x \colon t \to x(t; C)$ for all $t \in T \subset \mathbb{K}^m$ be the solutions to the completely solvable system (TDs).

Without loss of generality we'll consider that the first r rows of the functional matrix ${}^sX$ form the matrix of rank r (one can always get that by renumbering of dependent variables). Then, the first r components x_l, $l = 1, \ldots, r$, of the solutions are functionally independent on the domain T relative to variables t_ξ, $\xi = s+1, \ldots, m$, and the rest components x_ρ, $\rho = r + 1, \ldots, n$, of the solutions are functionally dependent on first r components on the domain T relative to variables t_ξ, $\xi = s + 1, \ldots, m$. So

$$x_\rho(t) = \Phi_\rho\bigl({}^st, {}^kx(t); C\bigr) \quad \text{for all } \ t \in T, \quad \rho = r + 1, \ldots, n,$$

where ${}^kx = (x_1, \ldots, x_k)$, and the functions $\Phi_\rho \colon T \to \mathbb{K}$, $\rho = r + 1, \ldots, n$, are holomorphic.

Taking into account the functional independence of functions x_l, $l = 1, \ldots, r$, relative to $t_{s+1}, \ldots, t_m$ on the domain T from the equalities $x_l = x_l(t; C)$, $l = 1, \ldots, r$, $x_\rho = \Phi_\rho({}^st, {}^kx; C)$, $\rho = r + 1, \ldots, n$, by fixation of arbitrary vector C by means of vectors $C^i = (\delta_{i1}C_1, \ldots, \delta_{in}C_n)$, $i = 1, \ldots, n$, where δ_{ij} is a Kronecker symbol, we find r not s-nonautonomous and $n-r$ s-nonautonomous functionally independent on the domain $\Pi' \subset \Pi$ first integrals of the system (TDs). ■

Let's pay attention to coordination relative to independent variables $t_1, \ldots, t_s$ between s-nonautonomy of system (TDs) and s-nonautonomy of first integrals in Theorem 1.4.

Theorem 1.5 and Theorem 1.6 are the corollaries of Theorem 1.4 for the autonomous total differential systems.

Theorem 1.5. *The completely solvable system* (ATD) *has exactly $n - r$, where $r = \operatorname{rank} X(x)$ for all $x \in G'$, functionally independent on the domain $G' \subset G$ autonomous first integrals.*

Theorem 1.6. *The completely solvable system* (ATD) *does not have the autonomous first integrals if and only if $\operatorname{rank} X(x) = n$ for all x from the domain G perhaps with the exception of point set of n-dimensional zero measure.*

Example 1.2. The completely solvable total differential system

$$dx_1 = dt_1, \quad dx_2 = dt_2, \quad dx_3 = \partial_{x_1} g(x_1, x_2)\, dt_1 + \partial_{x_2} g(x_1, x_2)\, dt_2, \tag{1.14}$$

where a holomorphic scalar function $g \colon \widetilde{G} \to \mathbb{K}$, $\widetilde{G} \subset \mathbb{K}^2$, has $n-r = 3-\operatorname{rank} X(x) = 3-2 = 1$ autonomous first integral $F \colon x \to g(x_1, x_2) - x_3$ for all $x \in G = \widetilde{G} \times \mathbb{K}$ (Theorem 1.5).

The functionally independent first integrals

$$F_1 \colon (t, x) \to t_1 - x_1 \ \text{ for all } \ (t, x) \in \Pi, \quad F_2 \colon (t, x) \to t_2 - x_2 \ \text{ for all } \ (t, x) \in \Pi,$$
$$F_3 \colon (t, x) \to g(x_1, x_2) - x_3 \ \text{ for all } \ (t, x) \in \Pi \tag{1.15}$$

forms an integral basis on the domain $\Pi = \mathbb{K}^3 \times \widetilde{G}$ of system (1.14).

1.3. Autonomy and cylindricality of first integrals for total differential system

Definition 1.4. *We'll say that a first integral* F *on a domain* $\Pi' \subset \Pi$ *of system* (TD) *is* $(n-k)$***-cylindrical*** *if the function* F *depends on* t *and only on* $k,\ 0 \leqslant k \leqslant n,$ *dependent variables* $x_1, \ldots, x_n$.

Let's define the problem of existence for system (TD) an s-nonautonomous $(n-k)$-cylindrical first integral

$$F\colon (t,x) \to F({}^s t, {}^k x) \quad \text{for all} \ \ (t,x) \in \Pi' \subset \Pi, \quad {}^s t = (t_1, \ldots, t_s), \quad {}^k x = (x_1, \ldots, x_k). \quad (1.16)$$

1.3.1. Necessary condition of existence of s-nonautonomous $(n-k)$-cylindrical first integral. According to the definition of first integral, the function (1.16) will be the first integral on the domain $\Pi' \subset \Pi$ of system (TD) if and only if

$$ {}^{sk}\mathfrak{X}_j F({}^s t, {}^k x) = 0 \quad \text{for all} \ \ (t,x) \in \Pi', \quad j = 1, \ldots, m, \qquad (1.17)$$

where the linear differential operators of first order

$$\begin{gathered} {}^{sk}\mathfrak{X}_\theta(t,x) = \partial_{t_\theta} + \sum_{\xi=1}^{k} X_{\xi\theta}(t,x)\partial_{x_\xi} \quad \text{for all} \ \ (t,x) \in \Pi, \quad \theta = 1, \ldots, s, \\ {}^{sk}\mathfrak{X}_\nu(t,x) = \sum_{\xi=1}^{k} X_{\xi\nu}(t,x)\partial_{x_\xi} \quad \text{for all} \ \ (t,x) \in \Pi, \quad \nu = s+1, \ldots, m. \end{gathered} \qquad (1.18)$$

Concerning the sets of functions ${}^k M_\theta = \{1, X_{1\theta}(t,x), \ldots, X_{k\theta}(t,x)\},\ \theta = 1, \ldots, s,\ {}^k M_\nu = \{X_{1\nu}(t,x), \ldots, X_{k\nu}(t,x)\},\ \nu = s+1, \ldots, m,$ the system of identeties (1.17) means that: the functions of each set ${}^k M_j,\ j = 1, \ldots, m,$ are linearly dependent with respect to independent variable t_ζ on the domain Π' under any fixed values of independent variables $t_\gamma,\ \gamma = 1, \ldots, m,\ \gamma \neq \zeta,$ and dependent variables $x_i,\ i = 1, \ldots, n;$ and the functions of each set ${}^k M_j,\ j = 1, \ldots, m,$ are linearly dependent with respect to dependent variable x_p on the domain Π' under any fixed values of independent variables $t_\gamma,\ \gamma = 1, \ldots, m,$ and dependent variables $x_i,\ i = 1, \ldots, n,\ i \neq p.$ It holds true under each fixed index $\zeta = s+1, \ldots, m$ and under each fixed index $p = k+1, \ldots, n.$

Therefore the Wronskians of each set ${}^k M_j,\ j = 1, \ldots, m,$ with respect to independent variables $t_\zeta,\ \zeta = s+1, \ldots, m,$ and dependent variables $x_p,\ p = k+1, \ldots, n$ vanish identically on the domain $\Pi',$ that is, the system of identities

$$\begin{gathered} \mathrm{W}_{t_\zeta}\big(1, {}^k X^\theta(t,x)\big) = 0 \quad \text{for all} \ \ (t,x) \in \Pi', \quad \theta = 1, \ldots, s,\ \zeta = s+1, \ldots, m, \\ \mathrm{W}_{t_\zeta}\big({}^k X^\nu(t,x)\big) = 0 \quad \text{for all} \ \ (t,x) \in \Pi', \quad \nu = s+1, \ldots, m,\ \zeta = s+1, \ldots, m, \\ \mathrm{W}_{x_p}\big(1, {}^k X^\theta(t,x)\big) = 0 \quad \text{for all} \ \ (t,x) \in \Pi', \quad \theta = 1, \ldots, s,\ p = k+1, \ldots, n, \\ \mathrm{W}_{x_p}\big({}^k X^\nu(t,x)\big) = 0 \quad \text{for all} \ \ (t,x) \in \Pi', \quad \nu = s+1, \ldots, m,\ p = k+1, \ldots, n, \end{gathered} \qquad (1.19)$$

is satisfied. Here vector functions ${}^k X^j\colon (t,x) \to \big(X_{1j}(t,x), \ldots, X_{kj}(t,x)\big)$ for all $(t,x) \in \Pi,$ $j = 1, \ldots, m,$ W_{t_ζ} and W_{x_p} are correspondingly the Wronskians with respect to t_ζ and $x_p,\ \zeta = s+1, \ldots, m,\ p = k+1, \ldots, n.$

So the *necessary test of existence of* s*-nonautonomous* $(n-k)$*-cylindrical first integral for total differential system* is proved.

Theorem 1.7. *The system of identities* (1.19) *is a necessary condition of existence of* s*-nonautonomous* $(n-k)$*-cylindrical first integral* (1.16) *for system* (TD).

1.3.2. Criterion of existence of s-nonautonomous $(n-k)$-cylindrical first integral. Let $n\times m$ matrix X of system (TD) satisfies the conditions (1.19). Let's write the functional system

$$
\begin{gathered}
\psi_\theta + {}^kX^\theta(t,x)\,{}^k\varphi = 0, \quad \theta=1,\dots,s,\\
\partial^\xi_{t_\zeta}\ {}^kX^\theta(t,x)\ {}^k\varphi = 0, \quad \theta=1,\dots,s,\ \zeta=s+1,\dots,m,\ \xi=1,\dots,k,\\
\partial^\xi_{x_p}\ {}^kX^\theta(t,x)\ {}^k\varphi = 0, \quad \theta=1,\dots,s,\ p=k+1,\dots,n,\ \xi=1,\dots,k,\\
{}^kX^\nu(t,x)\,{}^k\varphi = 0, \quad \nu=s+1,\dots,m,\\
\partial^\xi_{t_\zeta}\ {}^kX^\nu(t,x)\ {}^k\varphi = 0, \quad \nu=s+1,\dots,m,\ \zeta=s+1,\dots,m,\ \xi=1,\dots,k-1,\\
\partial^\xi_{x_p}\ {}^kX^\nu(t,x)\ {}^k\varphi = 0, \quad \nu=s+1,\dots,m,\ p=k+1,\dots,n,\ \xi=1,\dots,k-1,
\end{gathered}
\tag{1.20}
$$

where the vector functions ${}^s\psi\colon (t,x)\to\big(\psi_1({}^st,{}^kx),\dots,\psi_s({}^st,{}^kx)\big)$ for all $(t,x)\in\Pi'$ and ${}^k\varphi\colon (t,x)\to\big(\varphi_1({}^st,{}^kx),\dots,\varphi_k({}^st,{}^kx)\big)$ for all $(t,x)\in\Pi'$ are unknown, the vector functions ${}^kX^j\colon (t,x)\to\big(X_{1j}(t,x),\dots,X_{kj}(t,x)\big)$ for all $(t,x)\in\Pi$, $j=1,\dots,m$, $0\leqslant k\leqslant n$. Let's introduce a Pfaffian equation

$$
{}^s\psi({}^st,{}^kx)\,d\,{}^st + {}^k\varphi({}^st,{}^kx)\,d\,{}^kx = 0. \tag{1.21}
$$

Theorem 1.8 (*criterion of existence of s-nonautonomous $(n-k)$-cylindrical first integral for total differential system*). *For system* (TD) *to have s-nonautonomous $(n-k)$-cylindrical first integral* (1.16) *it is necessary and sufficient that there exist the vector functions ${}^s\psi$ and ${}^k\varphi$, satisfying functional system* (1.20), *such that the function* (1.16) *is the general integral of the Pfaffian equation* (1.21) *on the domain $\widetilde{\Pi}^{s+k}$ which is the natural projection of domain Π' on coordinate subspace $O\ {}^st\ {}^kx$.*

Proof. Necessity. Let system (TD) has the s-nonautonomous $(n-k)$-cylindrical first integral (1.16) on the domain Π'. Then, the identities (1.17) are satisfied:

$$
\partial_{t_\theta}F({}^st,{}^kx) + \sum_{\xi=1}^{k} X_{\xi\theta}(t,x)\partial_{x_\xi}F({}^st,{}^kx) = 0 \quad \text{for all } (t,x)\in\Pi', \quad \theta=1,\dots,s,
$$

$$
\sum_{\xi=1}^{k} X_{\xi\nu}(t,x)\partial_{x_\xi}F({}^st,{}^kx) = 0 \quad \text{for all } (t,x)\in\Pi', \quad \nu=s+1,\dots,m.
$$

By differentiating the first s of this identities k times with respect to $t_{s+1},\dots,t_m$ and k times with respect to $x_{k+1},\dots,x_n$ and by differentiating the rest $m-s$ identities $k-1$ times with respect to $t_{s+1},\dots,t_m$ and $k-1$ times with respect to $x_{k+1},\dots,x_n$ we conclude that the extensions on the domain Π' of the functions ${}^s\psi\colon ({}^st,{}^kx)\to\partial_{{}^st}F({}^st,{}^kx)$ for all $({}^st,{}^kx)\in\widetilde{\Pi}^{s+k}$ and ${}^k\varphi\colon ({}^st,{}^kx)\to\partial_{{}^kx}F({}^st,{}^kx)$ for all $({}^st,{}^kx)\in\widetilde{\Pi}^{s+k}$ is a solution to the functional system (1.20), where operators $\partial_{{}^st}=(\partial_{t_1},\dots,\partial_{t_s})$, $\partial_{{}^kx}=(\partial_{x_1},\dots,\partial_{x_k})$.

From this it also follows that the function (1.16) is a general integral on the domain $\widetilde{\Pi}^{s+k}$ of the Pfaffian equation (1.21).

Sufficiency. Let vector functions ${}^s\psi\colon (t,x)\to{}^s\psi({}^st,{}^kx)$, ${}^k\varphi\colon (t,x)\to{}^k\varphi({}^st,{}^kx)$ for all $(t,x)\in\Pi'$ be the solution to the system (1.20) and the Pfaffian equation (1.21) which is constructed on its base has the general integral (1.16) on the domain $\widetilde{\Pi}^{s+k}\subset\mathbb{K}^{s+k}$. Then, the system of identities

$$
\begin{gathered}
\partial_{{}^st}F({}^st,{}^kx) - \mu({}^st,{}^kx)\ {}^s\psi({}^st,{}^kx) = 0 \quad \forall({}^st,{}^kx)\in\widetilde{\Pi}^{s+k},\\
\partial_{{}^kx}F({}^st,{}^kx) - \mu({}^st,{}^kx)\ {}^k\varphi({}^st,{}^kx) = 0 \quad \forall({}^st,{}^kx)\in\widetilde{\Pi}^{s+k},
\end{gathered}
\tag{1.22}
$$

is satisfied, where $\mu\colon ({}^st, {}^kx) \to \mu({}^st, {}^kx)$ for all $({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}$ is a holomorphic integrating multiplier of the Pfaffian equation (1.21) which corresponds to its general integral (1.16) on the domain $\widetilde{\Pi}^{s+k}$.

Taking into account that the functions ${}^s\psi$, ${}^k\varphi$ are the solution to the functional system (1.20) we receive the system of identities (1.17). Therefore the function (1.16) is an s-nonautonomous $(n-k)$-cylindrical first integral on the dimain Π' of system (TD). ■

For example, the integral basis (1.15) of the autonomous completely solvable total differential system (1.14) (Example 1.2) consists of the autonomous first integral F_3 and two 1-nonautonomous 1-cylindrical first integrals F_1 and F_2.

Example 1.3. The autonomous total differential system

$$dx_1 = x_1\,dt_1 + 3x_1\,dt_2, \quad dx_2 = (1 + x_1 + 2x_2)\,dt_1 + (x_1 + 3x_2)\,dt_2 \tag{1.23}$$

is not completely solvable since the Poisson bracket

$$\big[\mathfrak{X}_1(t,x), \mathfrak{X}_2(t,x)\big] = (3 - x_1)\partial_{x_2} = \mathfrak{X}_{12}(t,x) \quad \text{for all } (t,x) \in \mathbb{K}^4$$

of induced by system (1.23) linear differential operators

$$\mathfrak{X}_1(t,x) = \partial_{t_1} + x_1\partial_{x_1} + (1 + x_1 + 2x_2)\partial_{x_2} \quad \text{for all } (t,x) \in \mathbb{K}^4,$$

$$\mathfrak{X}_2(t,x) = \partial_{t_2} + 3x_1\partial_{x_1} + (x_1 + 3x_2)\partial_{x_2} \quad \text{for all } (t,x) \in \mathbb{K}^4$$

is not the null operator on the any domain from the space $\mathbb{K}^4$.

By the Frobenius' theorem the system (1.23) doesn't have the solutions.

The associated to the system (1.23) incomplete normal linear homogeneous system of partial differential equations

$$\mathfrak{X}_1(t,x)y = 0, \quad \mathfrak{X}_2(t,x)y = 0$$

can be reduced to the complete system on the space $\mathbb{K}^4$ by the addition of single equation $\mathfrak{X}_{12}(t,x)y = 0$. Therefore this system has the defect $\delta = 1$ and an integral basis of system (1.23) consists of $n - \delta = 2 - 1 = 1$ first integral.

The system (1.23) is autonomous, but according to Theorem 1.4 it has no autonomous first integral. Indeed, the system of identities (1.19) are not satisfied, because, for example, the Wronskian

$$\mathrm{W}_{x_1}(1 + x_1 + 2x_2,\, x_1 + 3x_2) = \begin{vmatrix} 1 + x_1 + 2x_2 & x_1 + 3x_2 \\ 1 & 1 \end{vmatrix} = 1 - x_2 \quad \text{for all } (t,x) \in \mathbb{K}^4$$

does not vanish identically on the any domain from the space $\mathbb{K}^4$.

Let's find a 1-cylindrical first integral

$$F\colon (t,x) \to F(t,x_1) \quad \text{for all } (t,x) \in \Pi' \subset \mathbb{K}^4 \tag{1.24}$$

of system (1.23).

The Wronskians of the sets of functions ${}^1M_1 = \{1, x_1\}$ and ${}^1M_2 = \{1, 3x_1\}$ with respect to x_2 vanish identically on the space $\mathbb{K}^4$. Therefore the necessary conditions (Theorem 1.7) of existence of 1-cylindrical first integral (1.24) for system (1.23) are satisfied.

The functional system (1.20) consists of two equations

$$\psi_1 + x_1\,\varphi_1 = 0, \quad \psi_2 + 3x_1\,\varphi_1 = 0.$$

Its solution, for example, is

$$\psi_1\colon (t,x) \to x_1, \quad \psi_2\colon (t,x) \to 3x_1, \quad \varphi_1\colon (t,x) \to -1 \quad \text{for all } (t,x) \in \mathbb{K}^4.$$

The Pfaffian equation which is constructed on the base of this solution

$$x_1\,dt_1 + 3x_1\,dt_2 - dx_1 = 0$$

has the general integral $F\colon (t, x_1) \to x_1\, e^{-(t_1+3t_2)}$ for all $(t, x_1) \in \mathbb{K}^3$.

Therefore the system (1.23) on the space $\mathbb{K}^4$ has 1-cylindrical first integral

$$F\colon (t, x) \to x_1\, e^{-(t_1+3t_2)} \quad \text{for all } (t, x) \in \mathbb{K}^4.$$

This 1-cylindrical first integral forms the integral basis on the space $\mathbb{K}^4$ of system (1.23).

1.3.3. Functionally independent s-nonautonomous $(n-k)$-cylindrical first integrals.

Theorem 1.9. *Let the functional system* (1.20) *has q not linearly bound on the domain $\Pi' \subset \Pi$ solutions*

$$\begin{aligned} {}^s\psi^\gamma\colon (t, x) &\to {}^s\psi^\gamma({}^st, {}^kx) \quad \text{for all } (t, x) \in \Pi', \quad \gamma = 1, \dots, q, \\ {}^k\varphi^\gamma\colon (t, x) &\to {}^k\varphi^\gamma({}^st, {}^kx) \quad \text{for all } (t, x) \in \Pi', \quad \gamma = 1, \dots, q, \end{aligned} \tag{1.25}$$

and the Pfaffian equations

$${}^s\psi^\gamma({}^st, {}^kx)\; d\,{}^st + {}^k\varphi^\gamma({}^st, {}^kx)\; d\,{}^kx = 0, \quad \gamma = 1, \dots, q, \tag{1.26}$$

which are constructed on the base of this solutions have correspondingly general integrals

$$F_\gamma\colon ({}^st, {}^kx) \to F_\gamma({}^st, {}^kx) \quad \text{for all } ({}^st, {}^kx) \in \widetilde{\Pi}^{s+k} \subset \mathbb{K}^{s+k}, \quad \gamma = 1, \dots, q, \tag{1.27}$$

on the domain $\widetilde{\Pi}^{s+k}$ which is the natural projection of domain Π' on coordinate subspace $O\ {}^st\ {}^kx$. Then, this general integrals are functionally independent on the domain $\widetilde{\Pi}^{s+k}$.

Proof. By virtue of the system of identities (1.22)

$$\partial_{{}^st} F_\gamma({}^st, {}^kx) = \mu_\gamma({}^st, {}^kx)\ {}^s\psi^\gamma({}^st, {}^kx) \quad \text{for all } ({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}, \quad \gamma = 1, \dots, q,$$

$$\partial_{{}^kx} F_\gamma({}^st, {}^kx) = \mu_\gamma({}^st, {}^kx)\ {}^k\varphi^\gamma({}^st, {}^kx) \quad \text{for all } ({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}, \quad \gamma = 1, \dots, q.$$

Therefore the Jacobi's matrix

$$J\big(F_\gamma({}^st, {}^kx);\ {}^st, {}^kx\big) = \big\|\Psi({}^st, {}^kx)\Phi({}^st, {}^kx)\big\| \quad \text{for all } ({}^st, {}^kx) \in \widetilde{\Pi}^{s+k},$$

where the matrix $\|\Psi\Phi\|$ consists of $q \times s$ matrix $\Psi({}^st, {}^kx) = \big\|\mu_\gamma({}^st, {}^kx)\, \psi_j^\gamma({}^st, {}^kx)\big\|$ for all $({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}$ and $q \times k$ matrix $\Phi({}^st, {}^kx) = \big\|\mu_\gamma({}^st, {}^kx)\, \varphi_i^\gamma({}^st, {}^kx)\big\|$ for all $({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}$.

Since the vector functions (1.15) are not linearly bound on the domain $\widetilde{\Pi}^{s+k}$ the rank of Jacobi's matrix $\operatorname{rank} J\big(F_\gamma({}^st, {}^kx);\ {}^st, {}^kx\big) = q$ for all $({}^st, {}^kx)$ from the domain $\widetilde{\Pi}^{s+k}$ perhaps with the exception of point set of $(s+k)$-dimensional zero measure. So the general integrals (1.27) of the Pfaffian equations (1.26) are functionally independent on the domain $\widetilde{\Pi}^{s+k}$. ■

The Theorem 1.9 (taking into account the Theorem 1.8) let us to find a quantity of functionally independent s-nonautonomous $(n-k)$-cylindrical first integrals of system (TD).

2. Cylindricality and autonomy of last multipliers

2.1. Cylindricality of last multipliers for linear homogeneous system of partial differential equations

Definition 2.1. *We'll say that a last multiplier μ on a domain $G' \subset G$ of system (∂) is* ***$(n-k)$-cylindrical*** *if the function μ depends only on k, $0 \leqslant k \leqslant n$, variables $x_1, \dots, x_n$.*

Let's define the problem of existence for system (∂) an $(n-k)$-cylindrical last multiplier

$$\mu\colon x \to \mu({}^k x) \quad \text{for all} \;\; x \in G' \subset G, \quad {}^k x = (x_1, \ldots, x_k). \tag{2.1}$$

2.1.1. Necessary condition of existence of cylindrical last multiplier. According to the definition of last multiplier, the function (2.1) will be the last multiplier on the domain $G' \subset G$ of system (∂) if and only if

$$^k\mathfrak{L}_j \mu({}^k x) + \mu({}^k x) \operatorname{div} u^j(x) = 0 \quad \text{for all} \;\; x \in G', \quad j = 1, \ldots, m, \tag{2.2}$$

where the linear differential operators of first order ${}^k\mathfrak{L}_j,\ j = 1, \ldots, m,$ are defined by means of (1.3), the vector functions $u^j\colon x \to \big(u_{j1}(x), \ldots, u_{jn}(x)\big)$ for all $x \in G,\ j = 1, \ldots, m.$

The system of identities (2.2) in the coordinates is given by

$$\sum_{\xi=1}^{k} u_{j\xi}(x) \partial_{x_\xi} \mu({}^k x) + \mu({}^k x) \operatorname{div} u^j(x) = 0 \quad \text{for all} \;\; x \in G', \quad j = 1, \ldots, m. \tag{2.3}$$

Concerning the sets of functions ${}^kD_j = \big\{u_{j1}(x), \ldots, u_{jk}(x), \operatorname{div} u^j(x)\big\},\ j = 1, \ldots, m,$ the system of identeties (2.3) means that the functions of each set ${}^kD_j,\ j = 1, \ldots, m,$ are linearly dependent with respect to variable x_p on the domain G' under any fixed values of variables $x_i,\ i = 1, \ldots, n,\ i \ne p.$ It holds true under each fixed index $p = k+1, \ldots, n.$ Therefore the Wronskians of each set ${}^kD_j,\ j = 1, \ldots, m,$ with respect to variables $x_p,\ p = k+1, \ldots, n,$ vanish identically on the domain G', that is, the system of identities

$$\mathrm{W}_{x_p}\big({}^k u^j(x), \operatorname{div} u^j(x)\big) = 0 \quad \text{for all} \;\; x \in G', \quad j = 1, \ldots, m, \;\; p = k+1, \ldots, n, \tag{2.4}$$

is satisfied. Here the vector functions ${}^k u^j\colon x \to \big(u_{j1}(x), \ldots, u_{jk}(x)\big)$ for all $x \in G,\ j = 1, \ldots, m,$ and W_{x_p} are the Wronskians with respect to $x_p,\ p = k+1, \ldots, n.$

So the *necessary test of existence of* $(n-k)$*-cylindrical last multiplier for linear homogeneous system of partial differential equations* is proved.

Theorem 2.1. *The system of identities* (2.4) *is a necessary condition of existence of* $(n-k)$*-cylindrical last multiplier* (2.1) *for system* (∂).

2.1.2. Criterion of existence of cylindrical last multiplier. Let the $m \times n$ matrix $u(x) = \|u_{ji}(x)\|$ for all $x \in G$ of system (∂) satisfies the conditions (2.4). Let's write the functional system

$$\begin{gathered} {}^k u^j(x)\ {}^k\varphi = -\operatorname{div} u^j(x), \quad j = 1, \ldots, m, \\ \partial^{\xi}_{x_p}\ {}^k u^j(x)\ {}^k\varphi = -\partial^{\xi}_{x_p} \operatorname{div} u^j(x), \quad j = 1, \ldots, m, \;\; p = k+1, \ldots, n, \;\; \xi = 1, \ldots, k-1, \end{gathered} \tag{2.5}$$

where a vector function ${}^k\varphi\colon x \to \big(\varphi_1({}^k x), \ldots, \varphi_k({}^k x)\big)$ for all $x \in G'$ is unknown, the vector functions ${}^k u^j\colon x \to \big(u_{j1}(x), \ldots, u_{jk}(x)\big)$ for all $x \in G,\ j = 1, \ldots, m.$

Theorem 2.2 (*criterion of existence of* $(n-k)$*-cylindrical last multiplier for linear homogeneous system of partial differential equations*). *For system* (∂) *to have* $(n-k)$*-cylindrical last multiplier* (2.1) *it is necessary and sufficient that there exists a vector function* ${}^k\varphi$*, satisfying functional system* (2.5)*, such that the Pfaffian equation* (1.6) *which is constructed on the base of this vector function is exact on the domain* $\widetilde{G}^k$ *which is the natural projection of domain* G' *on coordinate subspace* $O\,{}^k x$*. At that, the last multiplier* (2.1) *of system* (∂) *is*

$$\mu\colon x \to \exp \int {}^k\varphi({}^k x)\ d\,{}^k x \quad \text{for all} \;\; x \in G'. \tag{2.6}$$

Proof. Necessity. Let system (∂) has the $(n-k)$-cylindrical last multiplier (2.1) on the

domain G'. Then, the system of identities (2.3) is satisfied. By means of termwise division of every identity (2.3) by $\mu({}^k x)$ we get a new system of identities

$$\sum_{\xi=1}^{k} u_{j\xi}(x)\partial_{x_\xi} \ln \mu({}^k x) + \operatorname{div} u^j(x) = 0 \quad \text{for all } \ x \in G_0' \subset G', \quad j = 1, \ldots, m.$$

By differentiating this identities $k-1$ times with respect to $x_p,\, p = k+1, \ldots, n,$ we conclude that an extension on the domain G_0' of the vector function

$$^k\varphi\colon {}^k x \to \big(\partial_{x_1} \ln \mu({}^k x), \ldots, \partial_{x_k} \ln \mu({}^k x)\big) \ \text{ for all } \ {}^k x \in \widetilde{G}_0^k \subset \mathbb{K}^k. \tag{2.7}$$

is a solution to the functional system (2.5).

The Pfaffian equation (1.6) which is constructed on the base of the vector function (2.7) is exact on the domain $\widetilde{G}_0^k$.

From (2.7) it follows that $(n-k)$-cylindrical last multiplier μ of system (∂) is constructing on the domain G_0' on the base of solutions to the system (2.5) by formula (2.6).

By restriction the domain G' to its codomain G_0' we conclude that the necessary condition of Theorem 2.2 is satisfied.

Sufficiency. Let the vector function $^k\varphi$ be a solution to the functional system (2.5) and the Pfaffian equation (1.6) which is constructed on its base is exact on the domain $\widetilde{G}^k \subset \mathbb{K}^k$. Then, the identities

$$\partial_{x_\xi} \int {}^k\varphi({}^k x)\ d\,{}^k x = \varphi_\xi({}^k x) \quad \text{for all } \ {}^k x \in \widetilde{G}^k, \quad \xi = 1, \ldots, k,$$

are satisfied.

Taking into account that the vector function $^k\varphi$ is a solution to the functional system (2.5) we receive the system of identities (2.2) for the function (2.6).

Hence, the function (2.6) is an $(n-k)$-cylindrical last miltiplier of system (∂). ■

Example 2.1. Consider the linear homogeneous system of partial differential equations

$$\mathfrak{L}_1(x)y = 0, \quad \mathfrak{L}_2(x)y = 0, \tag{2.8}$$

where the linear differential operators of first order

$$\mathfrak{L}_1(x) = x_1x_2\partial_{x_1} + x_1x_3\partial_{x_2} + x_1x_4\partial_{x_3} + x_2^2\,\partial_{x_4} \quad \text{for all } \ x \in \mathbb{R}^4,$$

$$\mathfrak{L}_2(x) = x_1x_3\partial_{x_1} + x_1x_4\partial_{x_2} + x_1^2\,\partial_{x_3} + x_2^2\,\partial_{x_4} \quad \text{for all } \ x \in \mathbb{R}^4.$$

Let's find for system (2.8) a 3-cylindrical last multiplier

$$\mu\colon x \to \mu(x_1) \quad \text{for all } \ x \in G' \subset \mathbb{R}^4. \tag{2.9}$$

The divergences

$$\operatorname{div} u^1(x) = \operatorname{div} \mathfrak{L}_1(x) = \partial_{x_1}(x_1x_2) + \partial_{x_2}(x_1x_3) + \partial_{x_3}(x_1x_4) + \partial_{x_4}x_2^2 = x_2 \quad \text{for all } \ x \in \mathbb{R}^4,$$

$$\operatorname{div} u^2(x) = \operatorname{div} \mathfrak{L}_2(x) = \partial_{x_1}(x_1x_3) + \partial_{x_2}(x_1x_4) + \partial_{x_3}x_1^2 + \partial_{x_4}x_2^2 = x_3 \quad \text{for all } \ x \in \mathbb{R}^4.$$

The Wronskians of the sets of functions $^1D_1 = \{x_1x_2, x_2\}$ and $^1D_2 = \{x_1x_3, x_3\}$ with respect to x_2, x_3, and x_4 vanish identically on the space $\mathbb{R}^4$:

$$\mathrm{W}_{x_2}(x_1x_2,\, x_2) = \begin{vmatrix} x_1x_2 & x_2 \\ x_1 & 1 \end{vmatrix} = 0, \quad \mathrm{W}_{x_3}(x_1x_2,\, x_2) = \mathrm{W}_{x_4}(x_1x_2,\, x_2) = 0 \quad \text{for all } \ x \in \mathbb{R}^4,$$

$$\mathrm{W}_{x_2}(x_1x_3,\, x_3) = 0, \quad \mathrm{W}_{x_3}(x_1x_3,\, x_3) = \begin{vmatrix} x_1x_3 & x_3 \\ x_1 & 1 \end{vmatrix} = 0, \quad \mathrm{W}_{x_4}(x_1x_3,\, x_3) = 0 \quad \text{for all} \ \ x \in \mathbb{R}^4.$$

Therefore the necessary conditions (Theorem 2.1) of existence of 3-cylindrical last multiplier (2.9) for system (2.8) are satisfied.

Let's write the functional system

$$x_1x_2\,\varphi_1 = \,-\,x_2, \quad x_1x_3\,\varphi_1 = \,-\,x_3, \quad x_1\,\varphi_1 = \,-\,1.$$

From this system we find $\varphi_1\colon x \to \,-\,1/x_1$ for all $x \in \mathbb{R}^4\backslash\{x\colon x_1 = 0\}$.

Since

$$\exp\int \frac{dx_1}{-\,x_1} = \frac{C}{|x_1|} \quad \text{for all} \ \ x_1 \in \mathbb{R}\backslash\{0\} \quad (C > 0),$$

the function $\mu\colon x \to -1/x_1$ for all $x \in G' \subset \mathbb{R}^4\backslash\{x\colon x_1 = 0\}$ is a 3-cylindrical last multiplier of system (2.8) (Theorem 2.2).

2.1.3. Functionally independent cylindrical last multipliers. The method which is proposed in Theorem 2.2 can be used to construct the functionally independent $(n-k)$-cylindrical last multipliers of system (∂).

Theorem 2.3. *Let the functional system* (2.5) *has* q *not linearly bound on the domain* $G' \subset G$ *solutions* (1.11) *and the corresponding Pfaffian equations* (1.12) *are exact on the domain* $\widetilde{G}^k$ *which is the natural projection of domain* G' *on coordinate subspace* $O\,{}^kx$*. Then, the* $(n-k)$*-cylindrical last multipliers of system* (∂)

$$\mu_\gamma\colon x \to \exp\int {}^k\varphi^\gamma({}^kx)\ d\,{}^kx \quad \text{for all} \ \ x \in G', \quad \gamma = 1, \ldots, q,$$

are functionally independent on the domain G'*.*

Proof. From Theorem 2.2 it follows that the last multipliers $\mu_\gamma,\, \gamma = 1, \ldots, q,$ of system (∂) are of indicated structure.

From representations

$$\partial_{x_\xi} \ln \mu_\gamma({}^kx) = \varphi^\gamma_\xi({}^kx) \quad \text{for all} \ \ {}^kx \in \widetilde{G}^k, \quad \xi = 1, \ldots, k, \ \gamma = 1, \ldots, q,$$

it follows that the Jacobi's matrix

$$J\bigl(\ln \mu_\gamma({}^kx);\, {}^kx\bigr) = \bigl\|\varphi^\gamma_\xi({}^kx)\bigr\|_{q\times k} \quad \text{for all} \ \ {}^kx \in \widetilde{G}^k.$$

Since the solutions (1.11) to the functional system (2.5) are not linearly bound on the domain G' the rank of Jacobi's matrix $\operatorname{rank} J\bigl(\ln \mu_\gamma({}^kx);\, {}^kx\bigr) = q$ nearly everywhere on the domain $\widetilde{G}^k$.

So the $(n-k)$-cylindrical last multipliers $\mu_\gamma,\, \gamma = 1, \ldots, q,$ of system (∂) are functionally independent on the domain G'. ■

2.2. Autonomy and cylindricality of last multipliers for total differential system

Definition 2.2. *We'll say that a last multiplier* μ *on a domain* $\Pi' \subset \Pi$ *of system* (TD) *is* ***s-nonautonomous*** *if the function* μ *depends on* x *and only on* $s,\, 0 \leqslant s \leqslant m,$ *independent variables* $t_1, \ldots, t_m$*. If* $s = 0,$ *then a last multiplier* $\mu\colon (t,x) \to \mu(x)$ *for all* $(t,x) \in \Pi'$ *of system* (TD) *is* ***autonomous****.*

Definition 2.3. *We'll say that a last multiplier* μ *on a domain* $\Pi' \subset \Pi$ *of system* (TD) *is* $\boldsymbol{(n-k)}$***-cylindrical*** *if the function* μ *depends on* t *and only on* $k,\, 0 \leqslant k \leqslant n,$ *dependent variables* $x_1, \ldots, x_n$*.*

Let's define the problem of existence for system (TD) an s-nonautonomous $(n-k)$-cylindrical last multiplier

$$\mu\colon (t,x)\to \mu({}^{s}t, {}^{k}x) \quad \text{for all } (t,x)\in \Pi'\subset \Pi, \quad {}^{s}t=(t_1,\ldots,t_s), \quad {}^{k}x=(x_1,\ldots,x_k). \qquad (2.10)$$

2.2.1. Necessary condition of existence of s-nonautonomous $(n-k)$-cylindrical last multiplier. According to the definition of last multiplier, the function (2.10) will be the last multiplier on the domain $\Pi'\subset\Pi$ of system (TD) if and only if

$$^{sk}\mathfrak{X}_j\mu({}^{s}t, {}^{k}x)+\mu({}^{s}t, {}^{k}x)\,\mathrm{div}_x X^j(t,x)=0 \quad \text{for all } (t,x)\in\Pi', \quad j=1,\ldots,m, \qquad (2.11)$$

where the linear differential operators of first order $^{sk}\mathfrak{X}_j,\ j=1,\ldots,m,$ are defined by means of (1.18), the vector functions $X^j\colon (t,x)\to \big(X_{1j}(t,x),\ldots,X_{nj}(t,x)\big)$ for all $(t,x)\in\Pi,\ j=1,\ldots,m,$ the divergence $\mathrm{div}_x X^j(t,x)=\sum\limits_{i=1}^{n}\partial_{x_i}X_{ij}(t,x)$ for all $(t,x)\in\Pi,\ j=1,\ldots,m.$

The system of identities (2.11) in the coordinates is given by

$$\partial_{t_\theta}\mu({}^{s}t, {}^{k}x)+\sum_{\xi=1}^{k} X_{\xi\theta}(t,x)\partial_{x_\xi}\mu({}^{s}t, {}^{k}x)+\mu({}^{s}t, {}^{k}x)\,\mathrm{div}_x X^\theta(t,x)=0 \quad \text{for all } (t,x)\in\Pi',$$

$$\sum_{\xi=1}^{k} X_{\xi\nu}(t,x)\partial_{x_\xi}\mu({}^{s}t, {}^{k}x)+\mu({}^{s}t, {}^{k}x)\,\mathrm{div}_x X^\nu(t,x)=0 \quad \text{for all } (t,x)\in\Pi', \qquad (2.12)$$

$$\theta=1,\ldots,s, \quad \nu=s+1,\ldots,m.$$

Concerning the sets of functions $^{k}B_\theta=\big\{1, X_{1\theta}(t,x),\ldots,X_{k\theta}(t,x),\ \mathrm{div}_x X^\theta(t,x)\big\},\ \theta=1,\ldots,s,$ $^{k}B_\nu=\big\{X_{1\nu}(t,x),\ldots,X_{k\nu}(t,x),\ \mathrm{div}_x X^\nu(t,x)\big\},\ \nu=s+1,\ldots,m,$ the system of identeties (2.12) means that: the functions of each set $^{k}B_j,\ j=1,\ldots,m,$ are linearly dependent with respect to independent variable t_ζ on the domain Π' under any fixed values of independent variables $t_\gamma,\ \gamma=1,\ldots,m,\ \gamma\ne\zeta,$ and dependent variables $x_i,\ i=1,\ldots,n;$ and the functions of each set $^{k}B_j,\ j=1,\ldots,m,$ are linearly dependent with respect to dependent variable x_p on the domain Π' under any fixed values of independent variables $t_\gamma,\ \gamma=1,\ldots,m,$ and dependent variables $x_i,\ i=1,\ldots,n,\ i\ne p.$ It holds true under each fixed index $\zeta=s+1,\ldots,m$ and under each fixed index $p=k+1,\ldots,n.$

Therefore the Wronskians of each set $^{k}B_j,\ j=1,\ldots,m,$ with respect to independent variables $t_\zeta,\ \zeta=s+1,\ldots,m,$ and dependent variables $x_p,\ p=k+1,\ldots,n$ vanish identically on the domain $\Pi',$ that is, the system of identities holds:

$$\begin{aligned}
&\mathrm{W}_{t_\zeta}\big(1, {}^{k}X^\theta(t,x),\ \mathrm{div}_x X^\theta(t,x)\big)=0 \ \text{ for all } (t,x)\in\Pi', \ \ \theta=1,\ldots,s,\ \zeta=s+1,\ldots,m,\\
&\mathrm{W}_{t_\zeta}\big({}^{k}X^\nu(t,x), \mathrm{div}_x X^\nu(t,x)\big)=0 \ \text{ for all } (t,x)\in\Pi', \ \ \nu=s+1,\ldots,m,\ \zeta=s+1,\ldots,m,\\
&\mathrm{W}_{x_p}\big(1, {}^{k}X^\theta(t,x),\ \mathrm{div}_x X^\theta(t,x)\big)=0 \ \text{ for all } (t,x)\in\Pi', \ \ \theta=1,\ldots,s,\ p=k+1,\ldots,n,\\
&\mathrm{W}_{x_p}\big({}^{k}X^\nu(t,x),\ \mathrm{div}_x X^\nu(t,x)\big)=0 \ \text{ for all } (t,x)\in\Pi', \ \ \nu=s+1,\ldots,m,\ p=k+1,\ldots,n,
\end{aligned} \qquad (2.13)$$

where the vector functions $^{k}X^j\colon (t,x)\to\big(X_{1j}(t,x),\ldots,X_{kj}(t,x)\big)$ for all $(t,x)\in\Pi,\ j=1,\ldots,m,$ W_{t_ζ} and W_{x_p} are correspondingly the Wronskians with respect to t_ζ and $x_p,\ \zeta=s+1,\ldots,m,\ p=k+1,\ldots,n.$

So the *necessary test of existence of s-nonautonomous $(n-k)$-cylindrical last multiplier for total differential system* is proved.

Theorem 2.4. *The system of identities* (2.13) *is a necessary condition of existence of* s*-nonautonomous* $(n-k)$*-cylindrical last multiplier* (2.10) *for system* (TD).

2.2.2. Criterion of existence of s-nonautonomous $(n-k)$-cylindrical last multiplier. Let the $n\times m$ matrix X of system (TD) satisfies the conditions (2.13). Let's write the functional system

$$\begin{gathered}
\psi_\theta + {}^kX^\theta(t,x)\ {}^k\varphi = -\operatorname{div}_x X^\theta(t,x), \quad \theta=1,\dots,s,\\
\partial^\xi_{t_\zeta}\ {}^kX^\theta(t,x)\ {}^k\varphi = -\partial^\xi_{t_\zeta}\operatorname{div}_x X^\theta(t,x), \quad \theta=1,\dots,s,\ \zeta=s+1,\dots,m,\ \xi=1,\dots,k+1,\\
\partial^\xi_{x_p}\ {}^kX^\theta(t,x)\ {}^k\varphi = -\partial^\xi_{x_p}\operatorname{div}_x X^\theta(t,x), \quad \theta=1,\dots,s,\ p=k+1,\dots,n,\ \xi=1,\dots,k+1,\\
{}^kX^\nu(t,x)\ {}^k\varphi = -\operatorname{div}_x X^\nu(t,x), \quad \nu=s+1,\dots,m, \qquad (2.14)\\
\partial^\xi_{t_\zeta}\ {}^kX^\nu(t,x)\ {}^k\varphi = -\partial^\xi_{t_\zeta}\operatorname{div}_x X^\nu(t,x), \quad \nu=s+1,\dots,m,\ \zeta=s+1,\dots,m,\ \xi=1,\dots,k,\\
\partial^\xi_{x_p}\ {}^kX^\nu(t,x)\ {}^k\varphi = -\partial^\xi_{x_p}\operatorname{div}_x X^\nu(t,x), \quad \nu=s+1,\dots,m,\ p=k+1,\dots,n,\ \xi=1,\dots,k,
\end{gathered}$$

where the vector functions ${}^s\psi\colon (t,x)\to \big(\psi_1({}^st,{}^kx),\dots,\psi_s({}^st,{}^kx)\big)$ for all $(t,x)\in\Pi'$ and ${}^k\varphi\colon (t,x)\to\big(\varphi_1({}^st,{}^kx),\dots,\varphi_k({}^st,{}^kx)\big)$ for all $(t,x)\in\Pi'$ are unknown, the vector functions $X^j\colon (t,x)\to\big(X_{1j}(t,x),\dots,X_{nj}(t,x)\big)$ for all $(t,x)\in\Pi$, $j=1,\dots,m$, and the vector functions ${}^kX^j\colon (t,x)\to\big(X_{1j}(t,x),\dots,X_{kj}(t,x)\big)$ for all $(t,x)\in\Pi$, $j=1,\dots,m$.

Theorem 1.8 (*criterion of existence of* s*-nonautonomous* $(n-k)$*-cylindrical last multiplier for total differential system*). *For system* (TD) *to have* s*-nonautonomous* $(n-k)$*-cylindrical last multiplier* (2.10) *it is necessary and sufficient that there exist the vector functions* ${}^s\psi$ *and* ${}^k\varphi$, *satisfying functional system* (2.14), *such that the Pfaffian equation* (1.21) *which is constructed on the base of this vector functions is exact on the domain* $\widetilde{\Pi}^{s+k}$ *which is the natural projection of domain* Π' *on coordinate subspace* $O\ {}^st\ {}^kx$. *At that, the last multiplier* (2.10) *of system* (TD) *is*

$$\mu\colon (t,x)\to \exp g({}^st,{}^kx) \quad \text{for all } (t,x)\in\Pi', \tag{2.15}$$

where

$$g\colon ({}^st,{}^kx)\to \int {}^s\psi({}^st,{}^kx)\ d\ {}^st + {}^k\varphi({}^st,{}^kx)\ d\ {}^kx \quad \text{for all } ({}^st,{}^kx)\in\widetilde{\Pi}^{s+k}. \tag{2.16}$$

Proof. Necessity. Let system (TD) has the s-nonautonomous $(n-k)$-cylindrical last multiplier (2.10) on the domain Π'. Then, the identities (2.12) are satisfied. By means of termwise division of every identity (2.12) by $\mu({}^st,{}^kx)$ we get a new system of identities

$$\partial_{t_\theta}\ln\mu({}^st,{}^kx)+\sum_{\xi=1}^{k}X_{\xi\theta}(t,x)\partial_{x_\xi}\ln\mu({}^st,{}^kx)+\operatorname{div}_xX^\theta(t,x)=0 \ \text{ for all } (t,x)\in\Pi'_0,\ \theta=1,\dots,s,$$

$$\sum_{\xi=1}^{k}X_{\xi\nu}(t,x)\partial_{x_\xi}\ln\mu({}^st,{}^kx)+\operatorname{div}_xX^\nu(t,x)=0 \quad \text{for all } (t,x)\in\Pi'_0\subset\Pi',\quad \nu=s+1,\dots,m.$$

By differentiating the first s of this identities k times with respect to $t_{s+1},\dots,t_m$ and k times with respect to $x_{k+1},\dots,x_n$ and by differentiating the rest $m-s$ identities $k-1$ times with respect to $t_{s+1},\dots,t_m$ and $k-1$ times with respect to $x_{k+1},\dots,x_n$ we conclude that the extensions on the domain Π'_0 of the functions

$$
\begin{aligned}
&{}^s\psi\colon ({}^st, {}^kx) \to \partial_{{}^st} \ln \mu({}^st, {}^kx) \quad \text{for all } \ ({}^st, {}^kx) \in \widetilde{\Pi}_0^{s+k} \subset \mathbb{K}^{s+k},\\
&{}^k\varphi\colon ({}^st, {}^kx) \to \partial_{{}^kx} \ln \mu({}^st, {}^kx) \quad \text{for all } \ ({}^st, {}^kx) \in \widetilde{\Pi}_0^{s+k},
\end{aligned}
\tag{2.17}
$$

is a solution to the functional system (2.14).

The Pfaffian equation (1.21) which is constructed on the base of the functions (2.17) is exact on the domain $\widetilde{\Pi}_0^{s+k}$.

From (2.17) it follows that s-nonautonomous $(n-k)$-cylindrical last multiplier μ of system (TD) is constructing on the domain Π_0' on the base of solutions to the functional system (2.14) by formula (2.15) with (2.16).

By restriction the domain Π' to its codomain Π_0' we conclude that the necessary condition of Theorem 2.5 is satisfied.

Sufficiency. Let the vector functions ${}^s\psi$ and ${}^k\varphi$ be a solution to the functional system (2.14) and the Pfaffian equation (1.21) which is constructed on its base is exact on the domain $\widetilde{\Pi}_0^{s+k} \subset \mathbb{K}^{s+k}$. Then,

$$
\partial_{{}^st}\, g({}^st, {}^kx) = {}^s\psi({}^st, {}^kx) \quad \text{for all } \ ({}^st, {}^kx) \in \widetilde{\Pi}^{s+k},
$$

$$
\partial_{{}^kx}\, g({}^st, {}^kx) = {}^k\varphi({}^st, {}^kx) \quad \text{for all } \ ({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}.
$$

Taking into account that the vector functions ${}^s\psi$ and ${}^k\varphi$ are a solution to the functional system (2.14) we receive that the system of identities (2.11) is satisfied ralative to the function (2.15) with (2.16).

Therefore the function (2.15) with (2.16) is an s-nonautonomous $(n-k)$-cylindrical last multiplier of system (TD). ■

2.2.3. Functionally independent s-nonautonomous $(n-k)$-cylindrical last multipliers. The method which is proposed in Theorem 2.5 can be used to construct the functionally independent s-nonautonomous $(n-k)$-cylindrical last multipliers of system (TD).

Theorem 2.6. *Let the functional system* (2.14) *has q not linearly bound on the domain Π' solutions* (1.25) *and the corresponding Pfaffian equations* (1.26) *are exact on the domain $\widetilde{\Pi}^{s+k}$ which is the natural projection of domain Π' on coordinate subspace $O\ {}^st\ {}^kx$. Then, the s-nonautonomous $(n-k)$-cylindrical last multipliers of system* (TD)

$$
\mu_\gamma\colon (t,x) \to \exp \int {}^s\psi^\gamma({}^st, {}^kx)\ d\ {}^st \,+\, {}^k\varphi^\gamma({}^st, {}^kx)\ d\ {}^kx \quad \text{for all } \ (t,x) \in \Pi', \quad \gamma = 1, \ldots, q,
$$

are functionally independent on the domain Π'.

Proof. From Theorem 2.5 it follows that the last multipliers $\mu_\gamma,\ \gamma = 1, \ldots, q,$ of system (TD) are of indicated structure.

From representations

$$
\partial_{t_\theta} \ln \mu_\gamma({}^st, {}^kx) = \psi_\theta^\gamma({}^st, {}^kx) \quad \text{for all } \ ({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}, \quad \theta = 1, \ldots, s, \ \ \gamma = 1, \ldots, q,
$$

$$
\partial_{x_\xi} \ln \mu_\gamma({}^st, {}^kx) = \varphi_\xi^\gamma({}^st, {}^kx) \quad \text{for all } \ ({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}, \quad \xi = 1, \ldots, k, \ \ \gamma = 1, \ldots, q,
$$

it follows that the Jacobi's matrix

$$
J\big(\ln \mu_\gamma({}^st, {}^kx); {}^st, {}^kx\big) = \big\| \Psi({}^st, {}^kx)\ \Phi({}^st, {}^kx) \big\|_{q \times (s+k)} \quad \text{for all } \ ({}^st, {}^kx) \in \widetilde{\Pi}^{s+k},
$$

where the matrix $\|\Psi\Phi\|$ consists of $q \times s$ matrix $\Psi({}^st, {}^kx) = \big\|\psi_\theta^\gamma({}^st, {}^kx)\big\|$ for all $({}^st, {}^kx) \in$ $\in \widetilde{\Pi}^{s+k}$ and $q \times k$ matrix $\Phi({}^st, {}^kx) = \big\|\varphi_\xi^\gamma({}^st, {}^kx)\big\|$ for all $({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}$.

Since the solutions (1.25) to the functional system (2.14) are not linearly bound on the

domain Π' the rank of Jacobi's matrix $\operatorname{rank} J\big(\ln \mu_\gamma({}^s t, {}^k x); {}^s t, {}^k x\big) = q$ nearly everywhere on the domain $\widetilde{\Pi}^{s+k}$. Therefore the s-nonautonomous $(n-k)$-cylindrical last multipliers μ_γ, $\gamma = 1,\ldots,q$, of system (TD) are functionally independent on the domain Π'. ■

3. Cylindricality and autonomy of partial integrals

3.1. Cylindricality of partial integrals for linear homogeneous system of partial differential equations

Definition 3.1. *We'll say that a partial integral w on a domain $G' \subset G$ of system (∂) is* ***$(n-k)$-cylindrical*** *if the function w depends only on k, $0 \leqslant k \leqslant n$, variables $x_1,\ldots,x_n$.*

Let's define the problem of existence for system (∂) an $(n-k)$-cylindrical partial integral

$$w\colon x \to w({}^k x) \quad \text{for all } x \in G' \subset G, \quad {}^k x = (x_1,\ldots,x_k). \tag{3.1}$$

3.1.1. Necessary condition of existence of cylindrical partial integral. According to the definition of partial integral, the function (3.1) will be the partial integral on the domain $G' \subset G$ of system (∂) if and only if

$${}^k\mathfrak{L}_j w({}^k x) = \Phi_j(x) \quad \text{for all } x \in G', \quad j = 1,\ldots,m, \tag{3.2}$$

where the linear differential operators of first order ${}^k\mathfrak{L}_j$, $j = 1,\ldots,m$, are defined by means of (1.3), the scalar functions $\Phi_j\colon G' \to \mathbb{K}$, $j = 1,\ldots,m$, are such that

$$\Phi_j(x)\big|_{w({}^k x)=0} = 0 \quad \text{for all } x \in G', \quad j = 1,\ldots,m. \tag{3.3}$$

The system of identities (3.2) in the coordinates is given by

$$\sum_{\xi=1}^{k} u_{j\xi}(x)\,\partial_{x_\xi} w({}^k x) = \Phi_j(x) \quad \text{for all } x \in G', \quad j = 1,\ldots,m. \tag{3.4}$$

Concerning the sets of functions ${}^k U_j = \big\{u_{j1}(x),\ldots,u_{jk}(x)\big\}$, $j = 1,\ldots,m$, the system of identeties (3.4) with (3.3) means that the functions of each set ${}^k U_j$, $j = 1,\ldots,m$, are linearly dependent with respect to variable x_p on the integral manifold $w({}^k x) = 0$ under any fixed values of variables x_i, $i = 1,\ldots,n$, $i \ne p$. It holds true under each fixed index $p = k+1,\ldots,n$. Therefore the Wronskians of each set ${}^k U_j$, $j = 1,\ldots,m$, with respect to variables x_p, $p = k+1,\ldots,n$, vanish identically on the integral manifold $w({}^k x) = 0$, that is, the system of identities holds:

$$\mathrm{W}_{x_p}\big({}^k u^j(x)\big) = \Xi_{jp}(x) \quad \text{for all } x \in G', \quad j = 1,\ldots,m, \ p = k+1,\ldots,n, \tag{3.5}$$

where the vector functions ${}^k u^j\colon x \to \big(u_{j1}(x),\ldots,u_{jk}(x)\big)$ for all $x \in G$, $j = 1,\ldots,m$, W_{x_p} are the Wronskians with respect to x_p, $p = k+1,\ldots,n$, the scalar functions $\Xi_{jp}\colon G' \to \mathbb{K}$, $j = 1,\ldots,m$, $p = k+1,\ldots,n$, are such that

$$\Xi_{jp}(x)\big|_{w({}^k x)=0} = 0 \quad \text{for all } x \in G', \quad j = 1,\ldots,m, \ p = k+1,\ldots,n. \tag{3.6}$$

So the *necessary test of existence of $(n-k)$-cylindrical partial integral for linear homogeneous system of partial differential equations* is proved.

Theorem 3.1. *The system of identities* (3.5) *with* (3.6) *is a necessary condition of existence of $(n-k)$-cylindrical partial integral* (3.1) *for system* (∂).

3.1.2. Criterion of existence of cylindrical partial integral. Let the $m \times n$ matrix $u(x) = \|u_{ji}(x)\|$ for all $x \in G$ of system (∂) satisfies the conditions $(3.5) \cup (3.6)$. Let's write

the functional system

$$\begin{aligned} &{}^k u^j(x)\,{}^k\varphi = H_j(x), \quad j=1,\dots,m, \\ &\partial^{\xi}_{x_p}\ {}^k u^j(x)\,{}^k\varphi = \partial^{\xi}_{x_p} H_j(x), \quad j=1,\dots,m, \ \xi=1,\dots,k-1, \ p=k+1,\dots,n, \end{aligned} \tag{3.7}$$

where a vector function ${}^k\varphi\colon x \to \bigl(\varphi_1({}^k x),\dots,\varphi_k({}^k x)\bigr)$ for all $x\in G'$ is unknown, the vector functions ${}^k u^j\colon x \to \bigl(u_{j1}(x),\dots,u_{jk}(x)\bigr)$ for all $x\in G,\ j=1,\dots,m,$ the scalar functions $H_j\colon G'\to\mathbb{K},\ j=1,\dots,m,$ are such that

$$H_j(x)\big|_{w({}^k x)=0} = 0 \quad \text{for all } \ x\in G', \quad j=1,\dots,m. \tag{3.8}$$

Theorem 3.2 (*criterion of existence of $(n-k)$-cylindrical partial integral for linear homogeneous system of partial differential equations*). *For system* (∂) *to have* $(n-k)$*-cylindrical partial integral* (3.1) *it is necessary and sufficient that there exists a vector function* ${}^k\varphi$ *and scalar functions* $H_j,\ j=1,\dots,m,$ *with* (3.8), *satisfying functional system* (3.7), *such that the Pfaffian equation* (1.6) *has the general integral* $w\colon {}^k x\to w({}^k x)$ *for all* ${}^k x\in \widetilde{G}^k$, *where domain* $\widetilde{G}^k$ *is the natural projection of domain* G' *on coordinate subspace* $O\,{}^k x$.

Proof. Necessity. Let system (∂) has the $(n-k)$-cylindrical partial integral (3.1) on the domain G'. Then, the system of identities (3.4) with (3.3) is satisfied. By differentiating this identities $k-1$ times with respect to $x_p,\ p=k+1,\dots,n,$ we conclude that an extension on the domain G' of the function

$${}^k\varphi\colon {}^k x\to \bigl(\partial_{x_1} w({}^k x),\dots,\partial_{x_k} w({}^k x)\bigr) \quad \text{for all } \ {}^k x\in\widetilde{G}^k$$

is a solution to the functional system (3.7) with (3.8). From this it also follows that the function (3.1) is the general integral on the domain $\widetilde{G}^k\subset\mathbb{K}^k$ of the Pfaffian equation (1.6).

Sufficiency. Let the vector function ${}^k\varphi\colon x\to {}^k\varphi({}^k x)$ for all $x\in G'$ be a solution to the functional system (3.7) with (3.8) and the Pfaffian equation (1.6) which is constructed on its base has the general integral $w\colon {}^k x\to w({}^k x)$ for all ${}^k x\in\widetilde{G}^k$. Then, the system of identities

$$\partial_{x_\xi} w({}^k x) - \mu({}^k x)\varphi_\xi({}^k x) = 0 \quad \text{for all } \ {}^k x\in\widetilde{G}^k, \quad \xi=1,\dots,k, \tag{3.9}$$

is satisfied. Here $\mu\colon {}^k x\to\mu({}^k x)$ for all ${}^k x\in\widetilde{G}^k$ is the holomorphic integrating multiplier of the Pfaffian equation (1.6) which corresponds to its general integral $w\colon {}^k x\to w({}^k x)$ for all ${}^k x\in\widetilde{G}^k$.

Taking into account that the vector function ${}^k\varphi$ is the solution to the functional system (3.7) with (3.8) we receive the system of identities (3.4), where

$$\Phi_j(x) = \mu({}^k x) H_j(x) \quad \text{for all } \ x\in G', \quad j=1,\dots,m.$$

Therefore the function (3.1) is an $(n-k)$-cylindrical partial integral on the domain G' of system (∂). ■

Example 3.1. Consider the linear homogeneous system of partial differential equations

$$\mathfrak{L}_1(x)y=0, \quad \mathfrak{L}_2(x)y=0, \tag{3.10}$$

where the linear differential operators of first order

$$\mathfrak{L}_1(x) = x_1(x_2+x_3)\partial_{x_1} + x_2(x_2+x_3)\partial_{x_2} + (x_1^2+x_2^2+x_3^2+x_4^2)\partial_{x_3} + (x_1^2-x_2^2+x_3^2-x_4^2)\partial_{x_4}$$
$$\text{for all } \ x\in\mathbb{K}^4,$$

$$\mathfrak{L}_2 = x_1(x_3+x_4)\partial_{x_1} + x_2(x_3+x_4)\partial_{x_2} + (x_1^2-x_2^2+x_3^2-x_4^2)\partial_{x_3} + (x_1^2+x_2^2+x_3^2+x_4^2)\partial_{x_4}$$
$$\text{for all } \ x\in\mathbb{K}^4.$$

Let's find for system (3.10) a 2-cylindrical partial integral

$$w\colon x \to w(x_1, x_2) \quad \text{for all} \ \ x \in G' \subset \mathbb{K}^4. \tag{3.11}$$

The Wronskians of the sets of functions ${}^2U_1 = \{x_1(x_2+x_3), x_2(x_2+x_3)\}$ and ${}^2U_2 = \{x_1(x_3+x_4), x_2(x_3+x_4)\}$ with respect to x_3 and x_4 vanish identically on the space $\mathbb{K}^4$:

$$\mathrm{W}_{x_3}\bigl(x_1(x_2+x_3), x_2(x_2+x_3)\bigr) = \begin{vmatrix} x_1(x_2+x_3) & x_2(x_2+x_3) \\ x_1 & x_2 \end{vmatrix} = 0 \quad \text{for all} \ \ x \in \mathbb{K}^4,$$

$$\mathrm{W}_{x_4}\bigl(x_1(x_2+x_3), x_2(x_2+x_3)\bigr) = 0 \quad \text{for all} \ \ x \in \mathbb{K}^4,$$

$$\mathrm{W}_{x_3}\bigl(x_1(x_3+x_4), x_2(x_3+x_4)\bigr) = \begin{vmatrix} x_1(x_3+x_4) & x_2(x_3+x_4) \\ x_1 & x_2 \end{vmatrix} = 0 \quad \text{for all} \ \ x \in \mathbb{K}^4,$$

$$\mathrm{W}_{x_4}\bigl(x_1(x_3+x_4), x_2(x_3+x_4)\bigr) = \begin{vmatrix} x_1(x_3+x_4) & x_2(x_3+x_4) \\ x_1 & x_2 \end{vmatrix} = 0 \quad \text{for all} \ \ x \in \mathbb{K}^4.$$

Therefore the necessary conditions (Theorem 3.1) of existence of 2-cylindrical partial integral (3.11) for system (3.10) are satisfied.

Let's write the functional system (3.7) with (3.8):

$$x_1(x_2+x_3)\,\varphi_1 + x_2(x_2+x_3)\,\varphi_2 = (x_1+x_2)(x_2+x_3), \quad x_1\,\varphi_1 + x_2\,\varphi_2 = x_1 + x_2,$$

$$x_1(x_3+x_4)\,\varphi_1 + x_2(x_3+x_4)\,\varphi_2 = (x_1+x_2)(x_3+x_4), \quad x_1\,\varphi_1 + x_2\,\varphi_2 = x_1 + x_2,$$

where $H_1(x) = (x_1+x_2)(x_2+x_3)$ for all $x \in \mathbb{K}^4$, $H_2(x) = (x_1+x_2)(x_3+x_4)$ for all $x \in \mathbb{K}^4$.

On the base of solution $\varphi_1\colon x \to 1$ for all $x \in \mathbb{K}^4$, $\varphi_2\colon x \to 1$ for all $x \in \mathbb{K}^4$ to this system we construct the Pfaffian equation

$$dx_1 + dx_2 = 0$$

which is exact (the integrating multiplier $\mu\colon (x_1, x_2) \to 1$ for all $(x_1, x_2) \in \mathbb{K}^2$) on the plane $\mathbb{K}^2$ and has the general integral $w\colon (x_1, x_2) \to x_1 + x_2$ for all $(x_1, x_2) \in \mathbb{K}^2$.

By extension of the general integral on the space $\mathbb{K}^4$ we get the 2-cylindrical partial integral $w\colon x \to x_1 + x_2$ for all $x \in \mathbb{K}^4$ of system (3.10).

3.1.3. Functionally independent cylindrical partial integral. The method which is proposed in Theorem 3.2 can be used to construct the functionally independent $(n-k)$-cylindrical partial integrals of system (∂).

Theorem 3.3. *Let h functional systems* (3.7) *with* (3.8) *has q not linearly bound on the domain $G' \subset G$ solutions* (1.11) *and for each of them the corresponding Pfaffian equation* (1.12) *has the general integral*

$$w_\gamma\colon {}^k x \to w_\gamma({}^k x) \quad \text{for all} \ \ {}^k x \in \widetilde{G}^k \subset \mathbb{K}^k, \quad \gamma = 1, \ldots, q, \tag{3.12}$$

on the domain $\widetilde{G}^k$ which is the natural projection of domain G' on coordinate subspace $O\,{}^k x$. Then, the general integrals (3.12) *are functionally independent on the domain $\widetilde{G}^k$.*

Proof. In accordance with the system of identities (3.9) we have

$$\partial_{x_\xi} w_\gamma({}^k x) - \mu_\gamma({}^k x)\varphi_\xi^\gamma({}^k x) = 0 \quad \text{for all} \ \ {}^k x \in \widetilde{G}^k, \quad \xi = 1, \ldots, k, \ \gamma = 1, \ldots, q.$$

Therefore the Jacobi's matrix $J\bigl(w_\gamma({}^k x);\, {}^k x\bigr) = \bigl\|\mu_\gamma({}^k x)\varphi_\xi^\gamma({}^k x)\bigr\|_{q\times k}$ for all ${}^k x \in \widetilde{G}^k$.

Since the vector functions (1.11) are not linearly bound on the domain $\widetilde{G}^k$ the rank

of Jacobi's matrix $\operatorname{rank} J\big(w_\gamma({}^k x);\, {}^k x\big) = q$ nearly everywhere on the domain $\widetilde{G}^k$. So the general integrals (3.12) of the Pfaffian equation (1.12) are functionally independent on the domain $\widetilde{G}^k$. ■

Example 3.2. The linear homogeneous system of partial differential equations

$$\mathfrak{L}_1(x)y = 0, \quad \mathfrak{L}_2(x)y = 0, \tag{3.13}$$

which is constructed on the base of linear differential operators of first order

$$\mathfrak{L}_1(x) = \sum_{i=1}^{5} x_i\,\partial_{x_i} \ \text{ for all } \ x \in \mathbb{K}^5, \qquad \mathfrak{L}_2(x) = \sum_{\nu=1}^{3} x_\nu\,\partial_{x_\nu} + x_4^2\,\partial_{x_4} + x_5^2\,\partial_{x_5} \ \text{ for all } \ x \in \mathbb{K}^5,$$

has the 4-cylindrical partial integrals

$$w_\nu\colon x \to x_\nu \ \text{ for all } \ x \in \mathbb{K}^5, \quad \nu = 1,2,3,$$

as $\mathfrak{L}_j x_\nu = x_\nu$ for all $x \in \mathbb{K}^5$, $j = 1,2$, $\nu = 1,2,3$.

Let's construct a basis of first integrals for system (3.13) on the base of this 4-cylindrical partial integrals.

The system (3.13) is incomplete and can be reduced to the complete system by the addition of single operator

$$\mathfrak{L}_{12}(x) = [\mathfrak{L}_1(x), \mathfrak{L}_2(x)] = x_4^2\,\partial_{x_4} + x_5^2\,\partial_{x_5} \ \text{ for all } \ x \in \mathbb{K}^5.$$

Therefore the incomplete system (3.13) has the defect $\delta = 1$ and its integral basis consists of $n - m - \delta = 5 - 2 - 1 = 2$ functionally independent first integrals.

Let's reduce the system $\mathfrak{L}_1(x)y = 0,\ \mathfrak{L}_2(x)y = 0,\ \mathfrak{L}_{12}(x)y = 0$ to the complete normal system

$$\partial_{x_1} y = -\,x_2 x_1^{-1}\,\partial_{x_2} y - x_3 x_1^{-1}\,\partial_{x_3} y, \quad \partial_{x_4} y = 0, \quad \partial_{x_5} y = 0$$

on a domain $H_1 \subset \{x\colon x_1 \neq 0\}$.

From this we find an integral basis on the domain H_1 of system (3.13), which consists of two functionally independent 3-cylindrical first integrals

$$F_{12}\colon x \to x_2\, x_1^{-1} \ \text{ for all } \ x \in H_1 \quad \text{ and } \quad F_{13}\colon x \to x_3\, x_1^{-1} \ \text{ for all } \ x \in H_1.$$

Similarly, the system $\mathfrak{L}_1(x)y = 0,\ \mathfrak{L}_2(x)y = 0,\ \mathfrak{L}_{12}(x)y = 0$ is normalized on the domains $H_\xi \subset \{x\colon x_\xi \neq 0\}$, $\xi = 2,3$, and the corresponding integral basises on the domains H_2 and H_3 of system (3.13) consist of functionally independent 3-cylindrical first integrals

$$F_{\xi\nu}\colon x \to x_\nu\, x_\xi^{-1} \ \text{ for all } \ x \in H_\xi, \qquad F_{\xi\theta}\colon x \to x_\theta\, x_\xi^{-1} \ \text{ for all } \ x \in H_\xi,$$

$$\xi = 2,3, \quad \nu = 1,2,3, \quad \nu \neq \xi, \quad \theta = 1,2,3, \quad \theta \neq \xi, \quad \nu \neq \theta.$$

3.2. Autonomy and cylindricality of partial integrals for total differential system

Definition 3.2. *We'll say that a partial integral w on a domain $\Pi' \subset \Pi$ of system* (TD) *is **s-nonautonomous** if the function w depends on x and only on s, $0 \leqslant s \leqslant m$, independent variables $t_1, \ldots, t_m$. If $s = 0$, then a partial integral $w\colon (t,x) \to w(x)$ for all $(t,x) \in \Pi'$ of system* (TD) *is **autonomous**.*

Definition 3.3. *We'll say that a partial integral w on a domain $\Pi' \subset \Pi$ of system* (TD) *is **$(n-k)$-cylindrical** if the function w depends on t and only on k, $0 \leqslant k \leqslant n$, dependent variables $x_1, \ldots, x_n$.*

Let's define the problem of existence for system (TD) an s-nonautonomous $(n-k)$-cylindrical partial integral

$$w\colon (t,x)\to w({}^st,{}^kx) \quad \text{for all } (t,x)\in \Pi'\subset \Pi, \quad {}^st=(t_1,\ldots,t_s), \quad {}^kx=(x_1,\ldots,x_k). \tag{3.14}$$

3.2.1. Necessary condition of existence of s-nonautonomous $(n-k)$-cylindrical partial integral. According to the definition of partial integral, the function (3.14) will be the partial integral on the domain $\Pi'\subset\Pi$ of system (TD) if and only if

$${}^{sk}\mathfrak{X}_j w({}^st,{}^kx)=\Phi_j(t,x) \quad \text{for all } (t,x)\in\Pi', \quad j=1,\ldots,m, \tag{3.15}$$

where the linear differential operators of first order ${}^{sk}\mathfrak{X}_j,\ j=1,\ldots,m,$ are defined by means of (1.18), the scalar functions $\Phi_j\colon \Pi'\to\mathbb{K},\ j=1,\ldots,m,$ are such that

$$\Phi_j(t,x)\big|_{w({}^st,{}^kx)=0}=0 \quad \text{for all } (t,x)\in\Pi', \quad j=1,\ldots,m. \tag{3.16}$$

The system of identities (3.15) in the coordinates is given by

$$\begin{gathered}\partial_{t_\theta} w({}^st,{}^kx)+\sum_{\xi=1}^{k} X_{\xi\theta}(t,x)\partial_{x_\xi} w({}^st,{}^kx)=\Phi_\theta(t,x) \quad \text{for all } (t,x)\in\Pi', \quad \theta=1,\ldots,s,\\ \sum_{\xi=1}^{k} X_{\xi\nu}(t,x)\partial_{x_\xi} w({}^st,{}^kx)=\Phi_\nu(t,x) \quad \text{for all } (t,x)\in\Pi', \quad \nu=s+1,\ldots,m.\end{gathered} \tag{3.17}$$

Concerning the sets of functions ${}^kM_\theta=\{1, X_{1\theta}(t,x),\ldots,X_{k\theta}(t,x)\},\ \theta=1,\ldots,s,\ {}^kM_\nu = \{X_{1\nu}(t,x),\ldots,X_{k\nu}(t,x)\},\ \nu=s+1,\ldots,m,$ the system of identeties (3.17) with (3.16) means that: the functions of each set ${}^kM_j,\ j=1,\ldots,m,$ are linearly dependent with respect to independent variable t_ζ on the integral manifold $w({}^st,{}^kx)=0$ under any fixed values of independent variables $t_\gamma,\ \gamma=1,\ldots,m,\ \gamma\neq\zeta,$ and dependent variables $x_i,\ i=1,\ldots,n;$ and the functions of each set ${}^kM_j,\ j=1,\ldots,m,$ are linearly dependent with respect to dependent variable x_p on the integral manifold $w({}^st,{}^kx)=0$ under any fixed values of independent variables $t_\gamma,\ \gamma=1,\ldots,m,$ and dependent variables $x_i,\ i=1,\ldots,n,\ i\neq p.$ It holds true under each fixed index $\zeta=s+1,\ldots,m$ and under each fixed index $p=k+1,\ldots,n.$

Therefore the Wronskians of each set ${}^kM_j,\ j=1,\ldots,m,$ with respect to independent variables $t_\zeta,\ \zeta=s+1,\ldots,m,$ and dependent variables $x_p,\ p=k+1,\ldots,n$ vanish identically on the integral manifold $w({}^st,{}^kx)=0,$ that is, the system of identities

$$\begin{gathered}\mathrm{W}_{t_\zeta}\big(1,{}^kX^\theta(t,x)\big)=\overset{*}{\Xi}_{\theta\zeta}(t,x) \quad \text{for all } (t,x)\in\Pi', \quad \theta=1,\ldots,s,\ \zeta=s+1,\ldots,m,\\ \mathrm{W}_{t_\zeta}\big({}^kX^\nu(t,x)\big)=\overset{*}{\Xi}_{\nu\zeta}(t,x) \quad \text{for all } (t,x)\in\Pi', \quad \nu=s+1,\ldots,m,\ \zeta=s+1,\ldots,m,\\ \mathrm{W}_{x_p}\big(1,{}^kX^\theta(t,x)\big)=\overset{**}{\Xi}_{\theta p}(t,x) \quad \text{for all } (t,x)\in\Pi', \quad \theta=1,\ldots,s,\ p=k+1,\ldots,n,\\ \mathrm{W}_{x_p}\big({}^kX^\nu(t,x)\big)=\overset{**}{\Xi}_{\nu p}(t,x) \quad \text{for all } (t,x)\in\Pi', \quad \nu=s+1,\ldots,m,\ p=k+1,\ldots,n,\end{gathered} \tag{3.18}$$

is satisfied. Here the vector functions ${}^kX^j\colon (t,x)\to\big(X_{1j}(t,x),\ldots,X_{kj}(t,x)\big)$ for all $(t,x)\in\Pi,\ j=1,\ldots,m,$ W_{t_ζ} and W_{x_p} are correspondingly the Wronskians with respect to t_ζ and $x_p,\ \zeta=s+1,\ldots,m,\ p=k+1,\ldots,n,$ the scalar functions $\overset{*}{\Xi}_{j\zeta}\colon\Pi'\to\mathbb{K},\ j=1,\ldots,m,\ \zeta=s+1,\ldots,m,$ and $\overset{**}{\Xi}_{jp}\colon\Pi'\to\mathbb{K},\ j=1,\ldots,m,\ p=k+1,\ldots,n,$ are such that

$$\begin{aligned}&\overset{*}{\Xi}_{j\zeta}(t,x)\big|_{w({}^st,{}^kx)=0}=0 \ \text{ for all } (t,x)\in\Pi', \quad j=1,\dots,m, \ \zeta=s+1,\dots,m,\\ &\overset{**}{\Xi}_{jp}(t,x)\big|_{w({}^st,{}^kx)=0}=0 \ \text{ for all } (t,x)\in\Pi', \quad j=1,\dots,m, \ p=k+1,\dots,n.\end{aligned} \tag{3.19}$$

So the *necessary test of existence of s-nonautonomous $(n-k)$-cylindrical partial integral for total differential system* is proved.

Theorem 3.4. *The system of identities* (3.18) *with* (3.19) *is a necessary condition of existence of the s-nonautonomous $(n-k)$-cylindrical partial integral* (3.14) *for system* (TD).

3.2.2. Criterion of existence of s-nonautonomous $(n-k)$-cylindrical partial integral. Let $n\times m$ matrix X of system (TD) satisfies the conditions (3.18) with (3.19). Let's write the functional system

$$\begin{gathered}\psi_\theta+{}^kX^\theta(t,x)\,{}^k\varphi=H_\theta(t,x), \quad \theta=1,\dots,s,\\ \partial^\xi_{t_\zeta}\ {}^kX^\theta(t,x)\ {}^k\varphi=\partial^\xi_{t_\zeta}H_\theta(t,x), \quad \theta=1,\dots,s, \ \zeta=s+1,\dots,m, \ \xi=1,\dots,k,\\ \partial^\xi_{x_p}\ {}^kX^\theta(t,x)\ {}^k\varphi=\partial^\xi_{x_p}H_\theta(t,x), \quad \theta=1,\dots,s, \ p=k+1,\dots,n, \ \xi=1,\dots,k,\\ {}^kX^\nu(t,x)\,{}^k\varphi=H_\nu(t,x), \quad \nu=s+1,\dots,m,\\ \partial^\xi_{t_\zeta}\ {}^kX^\nu(t,x)\ {}^k\varphi=\partial^\xi_{t_\zeta}H_\nu(t,x), \quad \nu=s+1,\dots,m, \ \zeta=s+1,\dots,m, \ \xi=1,\dots,k-1,\\ \partial^\xi_{x_p}\ {}^kX^\nu(t,x)\ {}^k\varphi=\partial^\xi_{x_p}H_\nu(t,x), \quad \nu=s+1,\dots,m, \ p=k+1,\dots,n, \ \xi=1,\dots,k-1,\end{gathered} \tag{3.20}$$

where the vector functions ${}^s\psi\colon (t,x)\to\big(\psi_1({}^st,{}^kx),\dots,\psi_s({}^st,{}^kx)\big)$ for all $(t,x)\in\Pi'$ and ${}^k\varphi\colon (t,x)\to\big(\varphi_1({}^st,{}^kx),\dots,\varphi_k({}^st,{}^kx)\big)$ for all $(t,x)\in\Pi'$ are unknown, the vector functions ${}^kX^j\colon (t,x)\to\big(X_{1j}(t,x),\dots,X_{kj}(t,x)\big)$ for all $(t,x)\in\Pi$, $j=1,\dots,m$, $0\leqslant k\leqslant n$, the scalar functions $H_j\colon \Pi'\to\mathbb{K}$, $j=1,\dots,m$, are such that

$$H_j(t,x)\big|_{w({}^st,{}^kx)=0}=0 \ \text{ for all } (t,x)\in\Pi', \quad j=1,\dots,m. \tag{3.21}$$

Theorem 3.5 (*criterion of existence of s-nonautonomous $(n-k)$-cylindrical partial integral for total differential system*). *For system* (TD) *to have s-nonautonomous $(n-k)$-cylindrical partial integral* (3.14) *it is necessary and sufficient that there exist the vector functions ${}^s\psi$, ${}^k\varphi$ and scalar functions H_j, $j=1,\dots,m$, with* (3.21), *satisfying functional system* (3.20), *such that the Pfaffian equation* (1.21) *has the general integral $w\colon ({}^st,{}^kx)\to w({}^st,{}^kx)$ for all $({}^st,{}^kx)\in\widetilde{\Pi}^{s+k}$, where the domain $\widetilde{\Pi}^{s+k}$ is the natural projection of domain Π' on coordinate subspace $O\ {}^st\ {}^kx$.*

Proof. Necessity. Let system (TD) has the s-nonautonomous $(n-k)$-cylindrical partial integral (3.14) on the domain Π'. Then, the identities (3.17) with (3.18) are satisfied. By differentiating the first s of this identities k times with respect to $t_{s+1},\dots,t_m$ and k times with respect to $x_{k+1},\dots,x_n$ and by differentiating the rest $m-s$ identities $k-1$ times with respect to $t_{s+1},\dots,t_m$ and $k-1$ times with respect to $x_{k+1},\dots,x_n$ we conclude that the extensions on the domain Π' of the functions ${}^s\psi\colon ({}^st,{}^kx)\to\big(\partial_{t_1}w({}^st,{}^kx),\dots,\partial_{t_s}w({}^st,{}^kx)\big)$ for all $({}^st,{}^kx)\in\widetilde{\Pi}^{s+k}$ and ${}^k\varphi\colon ({}^st,{}^kx)\to\big(\partial_{x_1}w({}^st,{}^kx),\dots,\partial_{x_k}w({}^st,{}^kx)\big)$ for all $({}^st,{}^kx)\in$ $\in\widetilde{\Pi}^{s+k}$ is a solution to the functional system (3.17) with (3.16). From this it also follows that the function (3.14) is a general integral on the domain $\widetilde{\Pi}^{s+k}\subset\mathbb{K}^{s+k}$ of the Pfaffian equation (1.21).

Sufficiency. Let the vector functions ${}^s\psi\colon (t,x)\to{}^s\psi({}^st,{}^kx)$, ${}^k\varphi\colon (t,x)\to{}^k\varphi({}^st,{}^kx)$ for all $(t,x)\in\Pi'$ be a solution to the functional system (3.20) with (3.21) and the Pfaffian equa-

tion (1.21) which is constructed on its base has the general integral $w\colon ({}^st, {}^kx) \to w({}^st, {}^kx)$ for all $({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}$. Then, the system of identities

$$\begin{aligned} &\partial_{t_\zeta} w({}^st, {}^kx) - \mu({}^st, {}^kx)\,\psi_\zeta({}^st, {}^kx) = 0 \quad \text{for all } ({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}, \ \ \zeta = 1, \ldots, s, \\ &\partial_{x_\xi} w({}^st, {}^kx) - \mu({}^st, {}^kx)\,\varphi_\xi({}^st, {}^kx) = 0 \quad \text{for all } ({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}, \ \ \xi = 1, \ldots, k, \end{aligned} \tag{3.22}$$

is satisfied, where $\mu\colon ({}^st, {}^kx) \to \mu({}^st, {}^kx)$ for all $({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}$ is a holomorphic along the manifold $w({}^st, {}^kx) = 0$ integrating multiplier of the Pfaffian equation (1.21) which corresponds to its general integral $w\colon ({}^st, {}^kx) \to w({}^st, {}^kx)$ for all $({}^st, {}^kx) \in \widetilde{\Pi}^{s+k}$.

Taking into account that the vector functions ${}^s\psi$, ${}^k\varphi$ are the solution to the functional system (3.20) with (3.21) we receive the system of identities (3.17) with $\Phi_j(t,x) = \mu({}^st, {}^kx)H_j(t,x)$ for all $(t,x) \in \Pi'$, $j = 1, \ldots, m$.

Therefore the function (3.14) is an s-nonautonomous $(n-k)$-cylindrical partial integral on the dimain Π' of system (TD). ■

Example 3.3. The real completely solvable autonomous total differential system

$$\begin{aligned} dx_1 &= -\big(x_2 + x_1(x_1^2 + x_2^2 + x_3^2)\big)\, dt_1 - x_1(x_1^2 + x_2^2 + x_3^2)\, dt_2, \\ dx_2 &= \big(x_1 - x_2(x_1^2 + x_2^2 + x_3^2)\big)\, dt_1 - x_2(x_1^2 + x_2^2 + x_3^2)\, dt_2, \\ dx_3 &= x_3(x_1^2 + x_2^2 + x_3^2)\,(dt_1 + dt_2) \end{aligned} \tag{3.23}$$

has the autonomous 2-cylindrical partial integral $w\colon (t,x) \to x_1^2 + x_2^2$ for all $(t,x) \in \mathbb{R}^5$. On the coordinate plane Ox_1x_2 of phase space $\mathbb{R}^3$ this partial integral specifies the isolated point $x_1 = x_2 = 0$ and for this point the hypotheses of Theorem 11 from [20] are satisfied, when

$$\partial_{t_1} w(t,x)\Big|_{\substack{(3.23)\\ x_3=0}} = \partial_{t_2} w(t,x)\Big|_{\substack{(3.23)\\ x_3=0}} = -\,2(x_1^2 + x_2^2)^2 \leqslant 0 \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2.$$

So the zero solution $x_1 = x_2 = x_3 = 0$ to system (3.23) is stable on the plane Ox_1x_2.

The equilibrium point $O(0,0,0)$ of the induced by system (3.23) autonomous ordinary differential system

$$\frac{dx_1}{dt_1} = -\,x_2 - x_1(x_1^2 + x_2^2 + x_3^2), \quad \frac{dx_2}{dt_1} = x_1 - x_2(x_1^2 + x_2^2 + x_3^2), \quad \frac{dx_3}{dt_1} = x_3(x_1^2 + x_2^2 + x_3^2)$$

is unstable by Chetaev's theorem [19, pp. 19 – 20] with $V(x_1, x_2, x_3) = -\,x_1^2 - x_2^2 + x_3^2$.

Therefore the zero solution $x_1 = x_2 = x_3 = 0$ to system (3.23) is unstable.

3.2.3. Functionally independent s-nonautonomous $(n-k)$-cylindrical partial integrals. The method which is proposed in Theorem 3.5 can be used to construct the functionally independent s-nonautonomous $(n-k)$-cylindrical partial integrals of system (TD).

Theorem 3.6. *Let h functional systems* (3.20) *with* (3.21) *has q not linearly bound on the domain $\Pi' \subset \Pi$ solutions* (1.25) *and for each of them the corresponding Pfaffian equation* (1.26) *has the general integral*

$$w_\gamma\colon ({}^st, {}^kx) \to w_\gamma({}^st, {}^kx) \quad \text{for all } ({}^st, {}^kx) \in \widetilde{\Pi}^{s+k} \subset \mathbb{K}^{s+k}, \quad \gamma = 1, \ldots, q, \tag{3.24}$$

on the domain $\widetilde{\Pi}^{s+k}$ which is the natural projection of domain Π' on coordinate subspace $O\,{}^st\,{}^kx$. Then, the general integrals (3.24) *are functionally independent on the domain $\widetilde{\Pi}^{s+k}$.*

Proof. By virtue of the system of identities (3.22)

$$\partial_{t_\zeta} w_\gamma({}^s t, {}^k x) - \mu_\gamma({}^s t, {}^k x)\,\psi_\zeta^\gamma({}^s t, {}^k x) = 0 \ \text{ for all } \ ({}^s t, {}^k x) \in \widetilde{\Pi}^{s+k}, \quad \zeta = 1, \dots, s, \ \gamma = 1, \dots, q,$$

$$\partial_{x_\xi} w_\gamma({}^s t, {}^k x) - \mu_\gamma({}^s t, {}^k x)\,\varphi_\xi^\gamma({}^s t, {}^k x) = 0 \ \text{ for all } \ ({}^s t, {}^k x) \in \widetilde{\Pi}^{s+k}, \quad \xi = 1, \dots, k, \ \gamma = 1, \dots, q.$$

So the Jacobi's matrix $J\big(w_\gamma({}^s t, {}^k x);\, {}^s t, {}^k x\big) = \big\|\Psi({}^s t, {}^k x)\Phi({}^s t, {}^k x)\big\|$ for all $({}^s t, {}^k x) \in \widetilde{\Pi}^{s+k}$, where the matrix $\|\Psi\Phi\|$ consists of $q \times s$ matrix $\Psi({}^s t, {}^k x) = \big\|\mu_\gamma({}^s t, {}^k x)\,\psi_\zeta^\gamma({}^s t, {}^k x)\big\|$ for all $({}^s t, {}^k x) \in \widetilde{\Pi}^{s+k}$ and $q \times k$ matrix $\Phi({}^s t, {}^k x) = \big\|\mu_\gamma({}^s t, {}^k x)\,\varphi_\xi^\gamma({}^s t, {}^k x)\big\|$ for all $({}^s t, {}^k x) \in \widetilde{\Pi}^{s+k}$.

Since the vector functions (1.25) are not linearly bound on the domain $\widetilde{\Pi}^{s+k}$ the rank of Jacobi's matrix $\operatorname{rank} J\big(w_\gamma({}^s t, {}^k x);\, {}^s t, {}^k x\big) = q$ nearly everywhere on the domain $\widetilde{\Pi}^{s+k}$. Therefore the general integrals (3.24) of the Pfaffian equations (1.26) are functionally independent on the domain $\widetilde{\Pi}^{s+k}$. ■

Example 3.4. The system of equations in total differentials

$$dx_i = x_i \left[\frac{t_2 - 1}{t_1(t_2 - t_1)}\, dt_1 - \frac{t_1 - 1}{t_2(t_2 - t_1)}\, dt_2\right], \quad i = 1, 2, 3, \tag{3.25}$$

is not completely solvable since the expression in square brackets is not the exact differential under independent variables t_1 and t_2.

The associated normal linear homogeneous partial system

$$\mathfrak{X}_1(t, x)\, y = 0, \quad \mathfrak{X}_2(t, x)\, y = 0,$$

which is constructed on the base of operators of differentiation by virtue of system (3.25)

$$\mathfrak{X}_1(t, x) = \partial_{t_1} + \frac{t_2 - 1}{t_1(t_2 - t_1)} \sum_{i=1}^{3} x_i\, \partial_{x_i} \quad \text{for all } \ (t, x) \in \Pi,$$

$$\mathfrak{X}_2(t, x) = \partial_{t_2} - \frac{t_1 - 1}{t_2(t_2 - t_1)} \sum_{i=1}^{3} x_i\, \partial_{x_i} \quad \text{for all } \ (t, x) \in \Pi,$$

is incomplete on every domain Π from the set $\{(t, x)\colon t_1 \neq 0,\ t_2 \neq 0,\ t_2 \neq t_1\}$ and has the defect $\delta = 1$.

Therefore an integral basis of system (3.25) consists of $n - \delta = 3 - 1 = 2$ functionally independent first integrals.

The system (3.25) has the autonomous 2-cylindrical partial integrals

$$w_i\colon (t, x) \to x_i \quad \text{for all } \ (t, x) \in \Pi, \quad i = 1, 2, 3,$$

since $\mathfrak{X}_1 x_i = \dfrac{t_2 - 1}{t_1(t_2 - t_1)}\, x_i$ for all $(t, x) \in \Pi$, $\mathfrak{X}_2 x_i = -\dfrac{t_1 - 1}{t_1(t_2 - t_1)}\, x_i$ for all $(t, x) \in \Pi$.

From system (3.25) we get

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \frac{dx_3}{x_3}\,.$$

From this by immediate integration we get that on every domain $\Pi_i \subset \{(t, x)\colon t_1 \neq 0,\ t_2 \neq 0,\ t_2 \neq t_1,\ x_i \neq 0\}$, $i = 1, 2, 3,$ the system (3.25) has the integral basis which consists of two functionally independent autonomous 1-cylindrical first integrals

$$F_{i\xi}\colon (t, x) \to x_\xi\, x_i^{-1} \quad \text{for all } \ (t, x) \in \Pi_i, \qquad F_{i\theta}\colon (t, x) \to x_\theta\, x_i^{-1} \quad \text{for all } \ x \in \Pi_i,$$

$$i = 1, 2, 3, \quad \xi = 1, 2, 3, \quad \theta = 1, 2, 3, \quad \xi \neq i, \quad \theta \neq i, \quad \theta \neq \xi.$$

3.3. Functional relations between general solutions to irreducible Painlevé equations

Let's consider the twelfth-order differential system

$$\frac{dx_i}{dt} = y_i, \quad \frac{dy_i}{dt} = P_i(t, x_i, y_i), \quad i = 1, \dots, 6, \tag{PS}$$

where

$$P_1\colon (t,x,y) \to 6x_1^2 + t \quad \text{for all } (t,x,y) \in \mathbb{C}^{13},$$

$$P_2\colon (t,x,y) \to 2x_2^3 + \alpha_2 + tx_2 \quad \text{for all } (t,x,y) \in \mathbb{C}^{13}, \quad \alpha_2 \in \mathbb{C},$$

$$P_3\colon (t,x,y) \to (-y_3 + \alpha_3 x_3^2 + \beta_3)t^{-1} + y_3^2 x_3^{-1} + \gamma_3 x_3^3 + \delta_3 x_3^{-1}$$

$$\text{for all } (t,x,y) \in \{(t,x,y)\colon t \neq 0,\ x_3 \neq 0\}, \quad \alpha_3, \beta_3, \gamma_3, \delta_3 \in \mathbb{C},$$

$$P_4\colon (t,x,y) \to \frac{1}{2}\ y_4^2 x_3^{-1} + \frac{3}{2}\ x_4^3 - 2\alpha_4 x_4 + \beta_4 x_4^{-1} + 4tx_4^2 + 2t^2 x_4$$

$$\text{for all } (t,x,y) \in \{(t,x,y)\colon x_4 \neq 0\}, \quad \alpha_4, \beta_4 \in \mathbb{C},$$

$$P_5\colon (t,x,y) \to (x_5-1)^2(\alpha_5 x_5 + \beta_5 x_5^{-1})t^{-2} + (-y_5 + \gamma_5 x_5)t^{-1} + \frac{1}{2}\ (3x_5 - 1)x_5^{-1}(x_5-1)^{-1}y_5^2 +$$

$$+ \delta_5 x_5(x_5+1)(x_5-1)^{-1} \text{ for all } (t,x,y) \in \{(t,x,y)\colon t \neq 0,\ x_5 \neq 0,\ x_5 \neq 1\},\ \alpha_5, \beta_5, \gamma_5, \delta_5 \in \mathbb{C},$$

$$P_6\colon (t,x,y) \to \Big(-\frac{1}{2}\ y_6^2 + y_6 - \delta_6\Big)(t-x_6)^{-1} + (x_6-1)^2(\alpha_6 x_6 + \beta_6 x_6^{-1})(t-1)^{-2} +$$

$$+ \big(-y_6 + \alpha_6 x_6(x_6-1)(1-2x_6) - \beta_6(x_6-1) + \gamma_6 x_6 + \delta_6 x_6\big)(t-1)^{-1} +$$

$$+ x_6^2\big(\alpha_6(x_6-1) - \gamma_6(x_6-1)^{-1}\big)t^{-2} + \big(-y_6 + \alpha_6 x_6(x_6-1)(2x_6-1) + \beta_6(x_6-1) - \gamma_6 x_6 -$$

$$- \delta_6(x_6-1)\big)t^{-1} + \frac{1}{2}\ \big(x_6^{-1} + (x_6-1)^{-1}\big)y_6^2$$

$$\text{for all } (t,x,y) \in \{(t,x,y)\colon t \neq 0,\ t \neq 1,\ x_6 \neq 0,\ x_6 \neq 1,\ x_6 \neq t\}, \quad \alpha_6, \beta_6, \gamma_6, \delta_6 \in \mathbb{C},$$

$$x = (x_1, \dots, x_6), \quad y = (y_1, \dots, y_6).$$

We'll consider the system (PS) on any domain Π from the set $D = \{(t,x,y)\colon t \neq 0,\ t \neq 1,\ x_3 \neq 0,\ x_4 \neq 0,\ x_5 \neq 0,\ x_5 \neq 1,\ x_6 \neq 0,\ x_6 \neq 1, x_6 \neq t\}$.

The system (PS) is constructed on the base of six irreducible Painlevé equations [21, pp. 463 – 465]

$$\frac{d^2x_1}{dt^2} = 6x_1^2 + t, \tag{P-1}$$

$$\frac{d^2x_2}{dt^2} = 2x_2^3 + tx_2 + \alpha_2\,, \tag{P-2}$$

$$\frac{d^2x_3}{dt^2} = \frac{1}{x_3}\Big(\frac{dx_3}{dt}\Big)^2 - \frac{1}{t}\ \frac{dx_3}{dt} + \frac{\alpha_3}{t}\ x_3^2 + \frac{\beta_3}{t} + \gamma_3 x_3^3 + \frac{\delta_3}{x_3}\,, \tag{P-3}$$

$$\frac{d^2x_4}{dt^2} = \frac{1}{2x_4}\Big(\frac{dx_4}{dt}\Big)^2 + \frac{3}{2}\ x_4^3 + 4tx_4^2 + 2(t^2-\alpha_4)x_4 + \frac{\beta_4}{x_4}\,, \tag{P-4}$$

$$\frac{d^2x_5}{dt^2} = \frac{3x_5-1}{2x_5(x_5-1)}\Big(\frac{dx_5}{dt}\Big)^2 - \frac{1}{t}\,\frac{dx_5}{dt} + \frac{\alpha_5}{t^2}\,x_5(x_5-1)^2 + \\ + \frac{\beta_5}{t^2}\,\frac{(x_5-1)^2}{x_5} + \frac{\gamma_5}{t}\,x_5 + \delta_5\,\frac{x_5(x_5+1)}{x_5-1}\,, \tag{P-5}$$

$$\frac{d^2x_6}{dt^2} = \frac{1}{2}\Big(\frac{1}{x_6} + \frac{1}{x_6-1} + \frac{1}{x_6-t}\Big)\Big(\frac{dx_6}{dt}\Big)^2 - \Big(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x_6-t}\Big)\frac{dx_6}{dt} + \\ + \frac{x_6(x_6-1)(x_6-t)}{t^2(t-1)^2}\,\Big(\alpha_6 + \beta_6\,\frac{t}{x_6^2} + \gamma_6\,\frac{t-1}{(x_6-1)^2} + \delta_6\,\frac{t(t-1)}{(x_6-t)^2}\Big). \tag{P-6}$$

Therefore we name the system (PS) the Painlevé system.

The components $x_i,\ i=1,\ldots,6,$ of the general solution to system (PS) are the general solutions to the irreducible Painlevé equations (P-i), $i=1,\ldots,6,$ correspondingly, and the components $y_i,\ i=1,\ldots,6,$ are their derivatives.

Thus, the problem on the relations between the general solutions to the irreducible Painlevé equations (P-i), $i=1,\ldots,6,$ and their derivatives

$$w(x_1,\ldots,x_6,\mathsf{D}x_1,\ldots,\mathsf{D}x_6)=0, \tag{3.26}$$

where $x_i,\ i=1,\ldots,6,$ are the general solutions to equations (P-i), $i=1,\ldots,6,$ correspondingly, is equivalent to the problem of finding autonomous partial integrals $w\colon (x,y)\to w(x,y)$ for all $(x,y)\in\widetilde{\Pi}^{12},$ where $\widetilde{\Pi}^{12}$ is the natural projection of domain Π on the phase space $\mathbb{C}^{12},$ for the differential Painlevé system (PS).

The existence of autonomous 6-cylindrical partial integrals $w\colon (x)\to w(x)$ for all $(x,y)\in\in\widetilde{\Pi}^{6},$ where $\widetilde{\Pi}^{6}$ is the natural projection of domain Π on the phase subspace $Ox,$ for the differential Painlevé system (PS) determines the relation w between the general solutions to the Painlevé equations (P-i), $i=1,\ldots,6.$ Otherwise, there are no relations given by holomorphic function w between the general solutions to the Painlevé equations (P-i), $i=1,\ldots,6.$

Let's seek autonomous partial integrals $w\colon (x,y)\to w(x,y)$ for all $(x,y)\in\widetilde{\Pi}^{12}$ of the Painlevé system (PS). To this end, we compose a system of the form (3.20) following Theorem 3.5 and consider the first equation of this system

$$\sum_{i=1}^{6} y_i\varphi_i + \sum_{i=7}^{12} P_{i-6}(t,x,y)\varphi_i = H_1(t,x,y),$$

where

$$H_1\colon (t,x,y)\to W_{-1}(x,y)(t-x_6)^{-1} + V_{-2}(x,y)(t-1)^{-2} + V_{-1}(x,y)(t-1)^{-1} + \\ + U_{-2}(x,y)t^{-2} + U_{-1}(x,y)t^{-1} + U_0(x,y) + U_1(x,y)t + U_2(x,y)t^2$$

and $W_{-1}(x,y)\big|_{w(x,y)=0}=0,\ V_{-2}(x,y)\big|_{w(x,y)=0}=0,\ V_{-1}(x,y)\big|_{w(x,y)=0}=0,\ U_j(x,y)\big|_{w(x,y)=0}= =0,\ j=-2,\ldots,2.$

Hence, by choosing

$$t^2,\quad t,\quad 1,\quad t^{-1},\quad t^{-2},\quad (t-1)^{-1},\quad (t-1)^{-2},\quad (t-x_6)^{-1}$$

as a basis, we obtain the system

$$\Big(-\frac{1}{2}\,y_6^2 + y_6 - \delta_6\Big)\varphi_{12} = W_{-1}(x,y),$$

$$(x_6-1)^2(\alpha_6x_6 + \beta_6x_6^{-1})\,\varphi_{12} = V_{-2}(x,y),$$

$$\big(-y_6+\alpha_6x_6(x_6-1)(1-2x_6)-\beta_6(x_6-1)+\gamma_6x_6+\delta_6x_6\big)\varphi_{12}=V_{-1}(x,y),$$

$$(x_5-1)^2(\alpha_5x_5+\beta_5x_5^{-1})\varphi_{11}+x_6^2\big(\alpha_6(x_6-1)-\gamma_6\big(x_6-1\big)^{-1}\big)\varphi_{12}=U_{-2}(x,y),$$

$$\big(-y_3+\alpha_3x_3^2+\beta_3\big)\varphi_9+\big(-y_5+\gamma_5x_5\big)\varphi_{11}+$$

$$+\big(-y_6+\alpha_6x_6(x_6-1)(2x_6-1)+\beta_6(x_6-1)-\gamma_6x_6-\delta_6(x_6-1)\big)\varphi_{12}=U_{-1}(x,y),$$

$$\sum_{i=1}^{6}y_i\varphi_i+6x_1^2\varphi_7+(2x_2^3+\alpha_2)\varphi_8+(y_3^2x_3^{-1}+\gamma_3x_3^3+\delta_3x_3^{-1})\varphi_9+ \tag{3.27}$$

$$+\Big(\frac{1}{2}\,y_4^2x_4^{-1}+\frac{3}{2}\,x_4^3-2\alpha_4x_4+\beta_4x_4^{-1}\Big)\varphi_{10}+$$

$$+\Big(\frac{1}{2}\,(3x_5-1)x_5^{-1}(x_5-1)^{-1}y_5^2+\delta_5x_5(x_5+1)(x_5-1)^{-1}\Big)\varphi_{11}+$$

$$+\Big(\frac{1}{2}\,(x_6^{-1}+(x_6-1)^{-1})y_6^2\Big)\varphi_{12}=U_0(x,y),$$

$$\varphi_7(x,y)+x_2\varphi_8+4x_4^2\varphi_{10}=U_1(x,y),$$

$$2x_4\varphi_{10}=U_2(x,y),$$

where the functions $\varphi_\xi\colon (x,y)\to\varphi_\xi(x,y)$ for all $(x,y)\in\widetilde{\Pi}^{12},\ \xi=1,\dots,12,$ are unknown.

Theorem 3.7. *There is no relation of the form* (3.26) *between the general solutions* $x_i\colon t\to x_i(t,C_{i1},C_{i2}),\ i=1,\dots,6,$ *to the irreducible Painlevé equations* (P-i), $i=1,\dots,6,$ *and their derivatives.*

Proof. Let $w\colon (t,x,y)\to w(x,y)$ for all $(t,x,y)\in\Pi$ be an autonomous nonconstant partial integral of the Painlevé system (PS).

It follows from the last equation of system (3.27) that

$$\varphi_{10}(x,y)=\frac{1}{2}\,x_4^{-1}\,U_2(x,y). \tag{3.28}$$

Let's consider the identity (see system (3.22))

$$\partial_{y_4}w(x,y)-\frac{1}{2}\,\mu(x,y)x_4^{-1}\,U_2(x,y)=0 \tag{3.29}$$

corresponding to the function φ_{10}. Since the integrating multiplier μ is holomorphic along $w(x,y)=0$ by Theorem 3.5, we get from identity (3.29) that w is a partial integral of the equation $x_4\,\partial_{y_4}w=0.$

From this it follows that

$$w=w(x,y_1,y_2,y_3,y_5,y_6), \tag{3.30}$$

that is, w is independent of y_4.

Since an autonomous partial integral is not identical constant, the multiplier μ don't vanish identically. Therefore,

$$\varphi_{10}(x,y)\equiv 0 \tag{3.31}$$

by virtue of (3.29) and (3.30) and on the base of representation (3.28).

From the first equation of system (3.27) we find

$$\varphi_{12}(x,y)=\Big(-\frac{1}{2}\,y_6^2+y_6-\delta_6\Big)^{-1}\,W_{-1}(x,y). \tag{3.32}$$

Similarly, considering the identity (see system (3.22))

$$\partial_{y_6} w(x,y) - \mu(x,y)\Big(- \frac{1}{2}\, y_6^2 + y_6 - \delta_6\Big)^{-1}\, W_{-1}(x,y) = 0, \tag{3.33}$$

corresponding to the function φ_{12}, we arrive at conclusion that w is a partial integral of the equation

$$\Big(- \frac{1}{2}\, y_6^2 + y_6 - \delta_6\Big)\partial_{y_6} w = 0.$$

From this taking into account (3.30) it follows that

$$w = w(x, y_1, y_2, y_3, y_5), \tag{3.34}$$

that is, w is independent of y_4 and y_6.

From (3.32), (3.33), and (3.34) it follows that

$$\varphi_{12}(x,y) \equiv 0. \tag{3.35}$$

In view of (3.31) and (3.35) the functional system (3.27) can be rewritten in the form

$$\begin{gathered}
(x_5-1)^2(\alpha_5 x_5 + \beta_5 x_5^{-1})\varphi_{11} = U_{-2}(x,y),\\
(- y_3 + \alpha_3 x_3^2 + \beta_3)\varphi_9 + (- y_5 + \gamma_5 x_5)\varphi_{11} = U_{-1}(x,y),\\
\sum_{i=1}^{6} y_i\varphi_i + 6x_1^2\varphi_7 + (2x_2^3 + \alpha_2)\varphi_8 + (y_3^2 x_3^{-1}\gamma_3 x_3^3 + \delta_3 x_3^{-1})\varphi_9 + \\
+ \Big(\frac{1}{2}\,(3x_5-1)x_5^{-1}(x_5-1)^{-1}y_5^2 + \delta_5 x_5(x_5+1)(x_5-1)^{-1}\Big)\varphi_{11} = U_0(x,y),\\
\varphi_7 + x_2\varphi_8 = U_1(x,y).
\end{gathered} \tag{3.36}$$

Let $|\alpha_5| + |\beta_5| \neq 0$. Then, from first equation of system (3.36) we get

$$\varphi_{11}(x,y) = (x_5-1)^{-2}\,(\alpha_5 x_5 + \beta_5 x_5^{-1})^{-1}\, U_{-2}(x,y). \tag{3.37}$$

The identity (see system (3.22))

$$\partial_{y_5} w(x,y) - \mu(x,y)(x_5-1)^{-2}(\alpha_5 x_5 + \beta_5 x_5^{-1})^{-1} U_{-2}(x,y) = 0 \tag{3.38}$$

corresponds to the function φ_{11} and by Theorem 3.5 a function w is a partial integral of the equation

$$(x_5-1)^2(\alpha_5 x_5 + \beta_5 x_5^{-1})\,\partial_{y_5} w = 0.$$

From this taking into account (3.34) it follows that

$$w = w(x, y_1, y_2, y_3), \tag{3.39}$$

that is, w is independent of y_4, y_5, y_6.

So, in view of (3.37), (3.38), and (3.39)

$$\varphi_{11}(x,y) \equiv 0. \tag{3.40}$$

From the second equation of system (3.36) with (3.40) it follows that

$$\varphi_9(x,y) = (- y_3 + \alpha_3 x_3^2 + \beta_3)^{-1}\, U_{-1}(x,y).$$

Next we consider the identity

$$\partial_{y_3} w(x,y) - \mu(x,y)(\,-y_3+\alpha_3 x_3^2+\beta_3)^{-1}\, U_{-1}(x,y)=0$$

and (by Theorem 3.5) taking into account (3.39) we establish that

$$w = w(x, y_1, y_2), \tag{3.41}$$

that is, w is independent of $y_\tau,\ \tau=3,\dots,6.$ Therefore,

$$\varphi_9(x,y)\equiv 0. \tag{3.42}$$

In view of (3.40) and (3.42) from the functional system (3.36) we get the system

$$\sum_{i=1}^{6} y_i\varphi_i + 6x_1^2\varphi_7 + (2x_2^3+\alpha_2)\varphi_8 = U_0(x,y), \qquad \varphi_7 + x_2\varphi_8 = U_1(x,y). \tag{3.43}$$

The system (3.43) has the solution

$$\varphi_i = z_i(x,y), \quad i=1,\dots,6,$$

$$\varphi_7 = (\,-6x_1^2x_2+2x_2^3+\alpha_2)^{-1}\Big(-x_2U_0(x,y)+(2x_2^3+\alpha_2)U_1(x,y)+x_2\sum_{i=1}^{6} y_iz_i(x,y)\Big), \tag{3.44}$$

$$\varphi_8=(\,-6x_1^2x_2+2x_2^3+\alpha_2)^{-1}\Big(U_0(x,y)-6x_1^2\,U_1(x,y)-\sum_{i=1}^{6} y_iz_i(x,y)\Big).$$

Taking into account (3.31), (3.35), (3.40), (3.42), and (3.44) the system (3.22) for $|\alpha_5|+|\beta_5|\neq 0$ assumes the form

$$\partial_{x_i} w(x,y) - \mu(x,y) z_i(x,y) = 0, \quad i=1,\dots,6,$$

$$\partial_{y_1} w(x,y) - \mu(x,y)(\,-6x_1^2x_2+2x_2^3+\alpha_2)^{-1}\Big(-x_2U_0(x,y)+$$

$$+(2x_2^3+\alpha_2)U_1(x,y)+x_2\sum_{i=1}^{6} y_iz_i(x,y)\Big)=0, \tag{3.45}$$

$$\partial_{y_2} w(x,y) - \mu(x,y)(\,-6x_1^2x_2+2x_2^3+\alpha_2)^{-1}\Big(U_0(x,y)-6x_1^2\,U_1(x,y)-\sum_{i=1}^{6} y_iz_i(x,y)\Big)=0,$$

$$\partial_{y_i} w(x,y)=0, \quad i=3,\dots,6,$$

From the seventh and the eighth identities of system (3.45) we get

$$\partial_{y_1} w(x,y) + x_2\,\partial_{y_2} w(x,y) = \mu(x,y)U_1(x,y)$$

and (by Theorem 3.5) a function w is a partial integral of equation

$$\partial_{y_1} w + x_2\,\partial_{y_2} w = 0.$$

Hence,

$$w = y_2 - y_1x_2 + h(x) \tag{3.46}$$

in view of representation (3.41).

From the first six identities and from the eighth identity of system (3.45) we get

$$\sum_{i=1}^{6} y_i\,\partial_{x_i} w(x,y) + (\,-6x_1^2x_2+2x_2^3+\alpha_2)\partial_{y_2} w(x,y) = \mu(x,y)\big(U_0(x,y)-6x_1^2U_1(x,y)\big).$$

Therefore, w is a partial integral of equation

$$\sum_{i=1}^{6} y_i\,\partial_{x_i}w + (\,-\,6x_1^2x_2 + 2x_2^3 + \alpha_2)\,\partial_{y_2}w = 0. \tag{3.47}$$

Since no function of the form (3.46) satisfies equation (3.47), we conclude that system (PS) doesn't have the autonomous partial integrals other then constants for $|\alpha_5| + |\beta_5| \neq 0.$

Let's consider the case $\alpha_5 = \beta_5 = 0.$

In this case system (3.36) has the general solution

$$\begin{gathered}
\varphi_i = z_i(x,y), \quad i = 1,\ldots,6,\\
\varphi_7 = \biggl((2x_2^3 + \alpha_2)U_1(x,y) + x_2\biggl(\sum_{i=1}^{6} y_i z_i(x,y) + (y_3^2x_3^{-1} + \gamma_3x_3^3 + \\
+\,\delta_3x_3^{-1})(\,-\,y_3 + \alpha_3x_3^2 + \beta_3)^{-1}\bigl(U_{-1}(x,y) - (\,-\,y_5 + \gamma_5x_5)g(x,y)\bigr) + \\
+\Bigl(\frac{1}{2}\,(3x_5 - 1)x_5^{-1}\,(x_5 - 1)^{-1}\,y_5^2 + \delta_5x_5(x_5+1)(x_5-1)^{-1}\Bigr)g(x,y) - \\
-\,U_0(x,y)\biggr)\biggr)(\,-\,6x_1^2x_2 + 2x_2^3 + \alpha_2)^{-1},\\
\varphi_8 = -\biggl(\sum_{i=1}^{6} y_iz_i(x,y) + 6x_1^2\,U_1(x,y) - (y_3^2x_3^{-1} + \gamma_3x_3^3 + \delta_3x_3^{-1})(\,-\,y_3 + \\
+\,\alpha_3x_3^2 + \beta_3)^{-1}\bigl(U_{-1}(x,y) - (\,-\,y_5 + \gamma_5x)g(x,y)\bigr) + \Bigl(\frac{1}{2}\,(3x_5 - 1)x_5^{-1}y_5^2 + \\
+\,\delta_5x_5(x_5+1)(x_5-1)^{-1}\Bigr)g(x,y) - U_0(x,y)\biggr)(\,-\,6x_1^2x_2 + 2x_2^3 + \alpha_2)^{-1},\\
\varphi_9 = \bigl(U_{-1}(x,y) - (\,-\,y_5 + \gamma_5x_5)g(x,y)\bigr)(\,-\,y_3 + \alpha_3x_3^3 + \beta_3)^{-1},\\
\varphi_{11} = g(x,y).
\end{gathered} \tag{3.48}$$

In view of (3.31), (3.35), and (3.48) system (3.22) for $\alpha_5 = \beta_5 = 0$ takes the form

$$\begin{gathered}
\partial_{x_i}w(x,y) - \mu(x,y)z_i(x,y) = 0, \quad i = 1,\ldots,6,\\
\partial_{y_1}w(x,y) - \mu(x,y)\varphi_7(x,y) = 0,\\
\partial_{y_2}w(x,y) - \mu(x,y)\varphi_8(x,y) = 0,\\
\partial_{y_3}w(x,y) - \mu(x,y)(\,-\,y_3 + \alpha_3x_3^2 + \beta_3)^{-1}\bigl(U_{-1}(x,y) - (\,-\,y_5 + \gamma_5x_5)g(x,y)\bigr) = 0,\\
\partial_{y_4}w(x,y) = 0,\\
\partial_{y_5}w(x,y) - \mu(x,y)g(x,y) = 0,\\
\partial_{y_6}w(x,y) = 0.
\end{gathered} \tag{3.49}$$

From the seventh and the eighth identities of system (3.49) we get

$$\partial_{y_1}w(x,y) + x_2\partial_{y_2}w(x,y) = \mu(x,y)U_1(x,y)$$

and much as we do in the first case we conclude that w is a partial integral of the equation $\partial_{y_1}w + x_2\,\partial_{y_2}w = 0.$

From this in view of (3.44) we obtain that

$$w = y_2 - y_1 x_2 + h(x, y_3, y_5). \tag{3.50}$$

It follows from the first six, eighth, and eleventh identities of system (3.49) that

$$\sum_{i=1}^{6} y_i \partial_{x_i} w(x,y) + (-6x_1^2 x_2 + \alpha_2)\partial_{y_2} w(x,y) +$$

$$+\biggl(\Bigl(\frac{1}{2}\,(3x_5-1)x_5^{-1}\,(x_5-1)^{-1}\,y_5^2 + \delta_5 x_5(x_5+1)(x_5-1)^{-1}\Bigr) -$$

$$-(-y_5+\gamma_5 x_5)(y_3^2 x_3^{-1} + \gamma_3 x_3^3 + \delta_3 x_3^{-1})(-y_3+\alpha_3 x_3^2+\beta_3)^{-1}\biggr)\partial_{y_5} w(x,y) =$$

$$= \mu(x,y)\bigl(-(y_3^2 x_3^{-1} + \gamma_3 x_3^3 + \delta_3 x_3^{-1})(-y_3+\alpha_3 x_3^2+\beta_3)^{-1}\,U_{-1}(x,y) + U_0(x,y) - 6x_1^2\,U_1(x,y)\bigr).$$

Therefore (by Theorem 3.5) a scalar function w is a partial integral of the equation

$$\sum_{i=1}^{6} y_i\,\partial_{x_i} w + (-6x_1^2 x_2 + 2x_2^3 + \alpha_2)\,\partial_{y_2} w +$$

$$+\biggl(\Bigl(\frac{1}{2}\,(3x_5-1)x_5^{-1}\,(x_5-1)^{-1}\,y_5^2 + \delta_5 x_5(x_5+1)(x_5-1)^{-1}\Bigr) - \tag{3.51}$$

$$-(-y_5+\gamma_5 x_5)(y_3^2 x_3^{-1} + \gamma_3 x_3^3 + \delta_3^{-1})(-y_3+\alpha_3 x_3^2+\beta_3)^{-1}\biggr)\partial_{y_5} w = 0.$$

Since no function of the form (3.50) satisfies equation (3.51), the Painlevé system (PS) doesn't have the autonomous partial integrals other then constants for $\alpha_5 = \beta_5 = 0.$ ■

Thus, we have given answers to the questions concerning the relations between the general solutions to the irreducible Painlevé equations (P-i), $i = 1,\ldots,6.$ These answers are as follows.

1. There is no functional relation $w(x_1,\ldots,x_6) = 0$ with holomorphic function w between the general solutions $x_i,\ i = 1,\ldots,6,$ to the Painlevé equations (P-i), $i = 1,\ldots,6.$

2. There is no functional relation of the form (3.26) with holomorphic function w between the general solutions $x_i,\ i = 1,\ldots,6,$ to the Painlevé equations (P-i), $i = 1,\ldots,6,$ and their derivatives $\mathsf{D}x_i.$

At the same time, the differential Painlevé system (PS) has nonautonomous partial integrals and therefore we can assert as follows.

3. There exists a functional relation of the form $w(t, x_1,\ldots,x_6,\ \mathsf{D}x_1,\ldots,\mathsf{D}x_6) = 0$ between the general solutions $x_i,\ i = 1,\ldots,6,$ to the Painlevé equations (P-i), $i = 1,\ldots,6,$ and their first derivatives $\mathsf{D}x_i.$

In particular, if the general solution $x_k,\ k\in\{1,\ldots,6\},$ to the k-th Painlevé equation (P-k) and its derivative $\mathsf{D}x_k$ are known, then we can assert as follows.

4. There exists a first-order ordinary differential equation of the form

$$w_s(t, z, \mathsf{D}z, x_k, \mathsf{D}x_k) = 0,$$

where t is the independent variable, z is an unknown function, x_k is the general solution to the Painlevé equation (P-k), whose general solution is the general solution to the s-th Painlevé equation (P-s), $s\in\{1,\ldots,6\},\ s\neq k,$ that is, $z = x_s,\ s\neq k.$

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